Circulation and Vorticity

Ramesh Vellore

Dynamics of Atmosphere and Ocean

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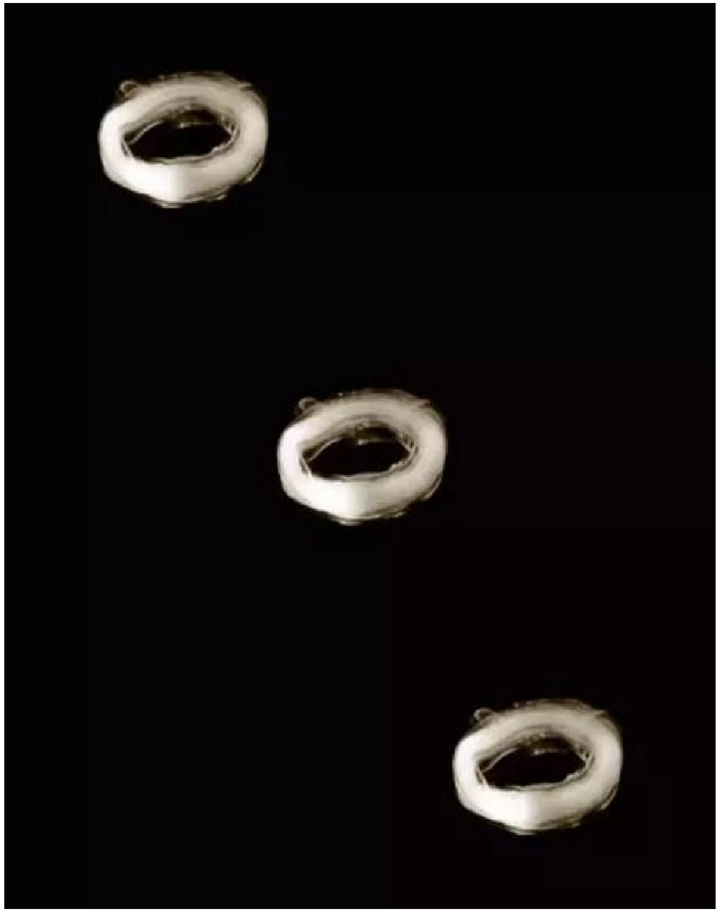
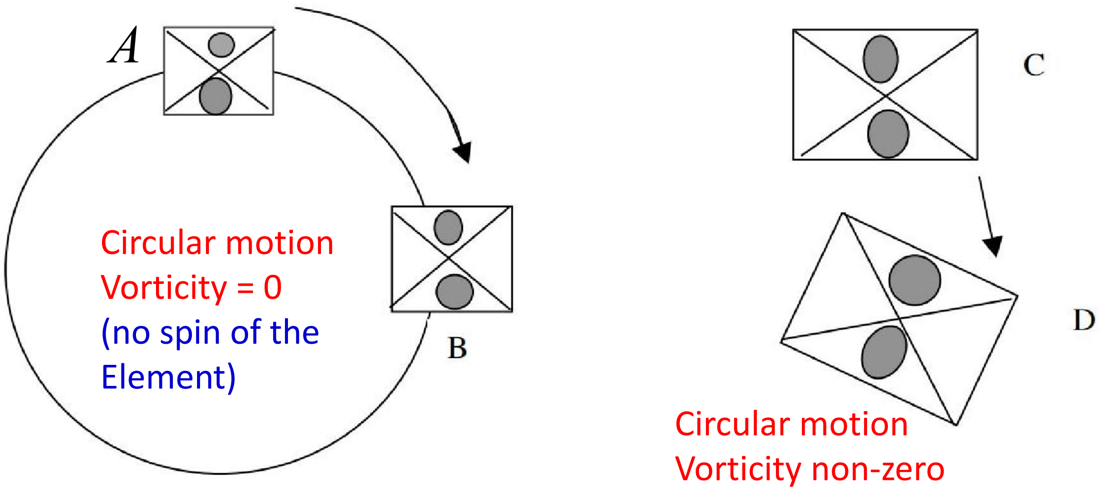




The fluid element moving from A to B on a circular path has no vorticity while the fluid element moving from C to D has vorticity.

Vorticity and circulation

There are two complementary concepts that quantify the rotational property of a flow :

Vorticity and circulation.

For solid objects we do not speak of the vorticity of an object but instead we refer to its angular velocity.

Vorticity is a local measure, whereas circulation is a bulk (integral) measure. Each can be defined either in the Lagrangian sense or in the Eulerian sense.

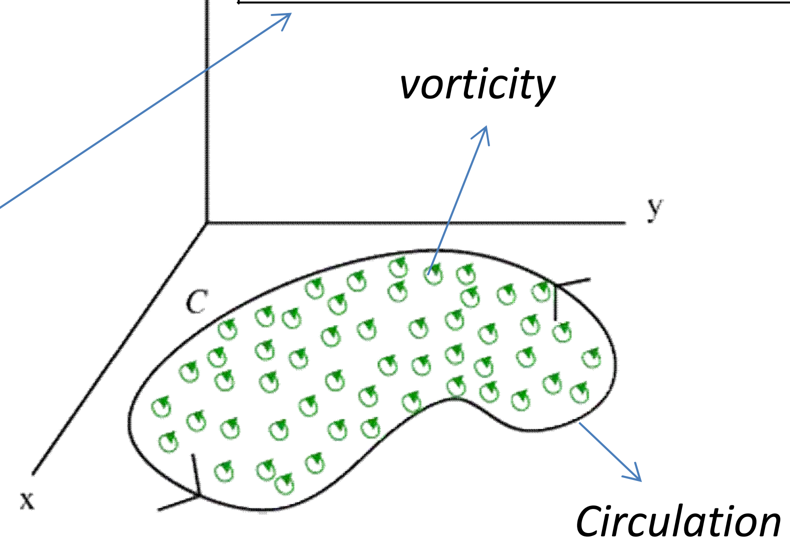
Vorticity ($\vec{\boldsymbol{\omega}}$) is defined as the curl of the velocity :

$$\vec{\boldsymbol{\omega}} = \nabla \times \vec{\mathbf{V}} = \nabla \times \left(u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}} \right)$$

$$\vec{\boldsymbol{\omega}} = \nabla \times \vec{\mathbf{V}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \xi\hat{\mathbf{i}} + \eta\hat{\mathbf{j}} + \zeta\hat{\mathbf{k}}$$

$$\xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right); \quad \eta = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right); \quad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Any velocity vector is intrinsically orthogonal to the corresponding vorticity vector



$$\int_C \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}} = \iint_A \left(\nabla \times \vec{\mathbf{V}} \right) \cdot \hat{\mathbf{n}} \, dA = \iint_A \vec{\boldsymbol{\omega}} \cdot \hat{\mathbf{n}} \, dA$$

$$\hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{V}} \right) = \zeta = \lim_{A \to 0} \oint \frac{\vec{\mathbf{V}} \cdot d\vec{\mathbf{l}}}{A}$$

Stokes's theorem: the circulation around a contour that contains a group of vortices is just equal to the sum of the enclosed vortex strengths.

Circulation and vorticity on sphere

Large-scale: the amount of (net) force that pushes along a closed boundary or path

Local scale: measure of the angular velocity of the fluid

One can compute the relative circulation (Γ_r) of a flow field on sphere:

$$\Gamma_r = v_\lambda(\varphi)\, a\cos\varphi\,\Delta\lambda + v_\varphi(\lambda+\Delta\lambda)\, a\Delta\varphi - v_\lambda(\varphi+\Delta\varphi)\, a\cos(\varphi+\Delta\varphi)\Delta\lambda - v_\varphi(\lambda)\, a\Delta\varphi$$

$$\frac{\Gamma_r}{\Delta\lambda\Delta\varphi} = \frac{v_\lambda(\varphi)\, a\cos\varphi}{\Delta\varphi} + \frac{v_\varphi(\lambda+\Delta\lambda)\, a}{\Delta\lambda} - \frac{v_\lambda(\varphi+\Delta\varphi)\, a\cos(\varphi+\Delta\varphi)}{\Delta\varphi} - \frac{v_\varphi(\lambda)\, a}{\Delta\lambda}$$

$$\frac{\Gamma_r}{\Delta\lambda\Delta\varphi} = a\left(\frac{v_\varphi(\lambda+\Delta\lambda) - v_\varphi(\lambda)}{\Delta\lambda}\right) - a\left(\frac{v_\lambda(\varphi+\Delta\varphi)\cos(\varphi+\Delta\varphi) - v_\lambda(\varphi)\cos\varphi}{\Delta\varphi}\right)$$

For small $\Delta\lambda, \Delta\varphi$ $\quad \{a \equiv$ radius of the Earth$\}$

$$\boxed{\frac{\Gamma_r}{\Delta\lambda\Delta\varphi} = a\left[\frac{\partial v_\varphi}{\partial\lambda} - \frac{\partial}{\partial\varphi}(v_\lambda\cos\varphi)\right]}$$

Recall that by Stokes' theorem, $\Gamma_r = \zeta_r A$ **[Circulation has the units of $m^2 s^{-1}$]**

ζ_r = vertical component of relative vorticity

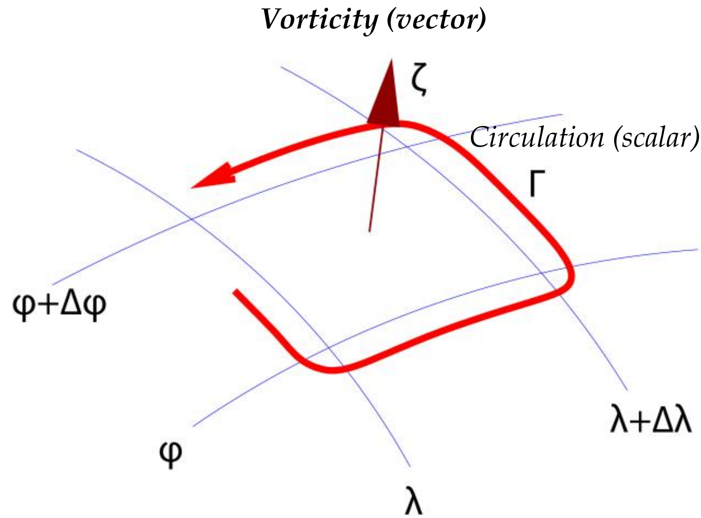
where $A = \Delta x_\lambda \Delta x_\varphi = (a\cos\varphi\,\Delta\lambda)(a\,\Delta\varphi) = a^2\cos\varphi\,\Delta\lambda\Delta\varphi$

$$\boxed{\zeta_{\text{rel}}^{\text{sphere}} = \frac{1}{a\cos\varphi}\left[\frac{\partial v_\varphi}{\partial\lambda} - \frac{\partial}{\partial\varphi}(v_\lambda\cos\varphi)\right]}$$

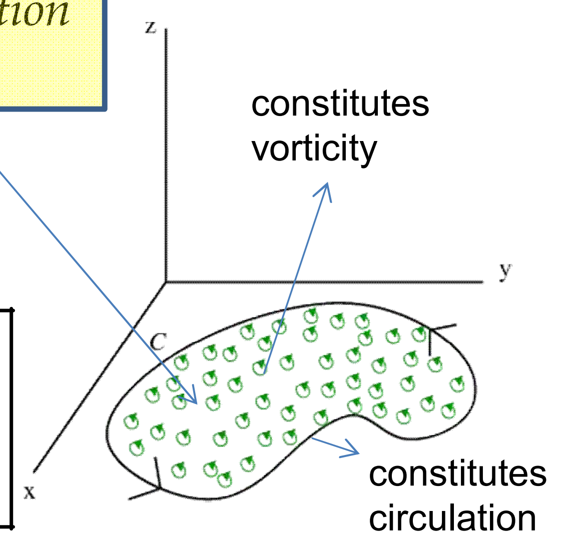
Absolute vorticity = Relative vorticity + Planetary vorticity

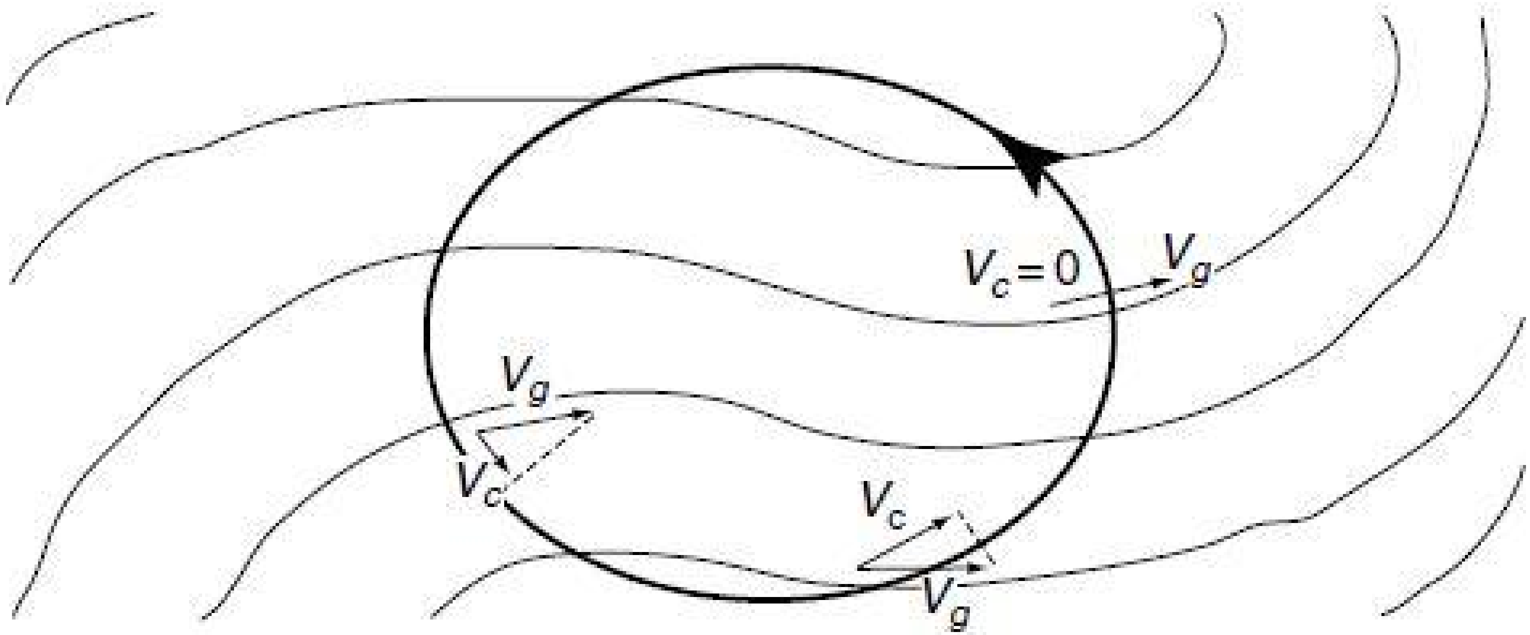
Conservation of absolute vorticity : $\zeta_a = \zeta_r + f = \zeta_r + 2\Omega\sin\varphi = \text{constant}$

$$\boxed{\zeta_{\text{abs}}^{\text{sphere}} = \frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)\right] + 2\Omega\sin\varphi}$$



Vorticity = Circulation per unit area



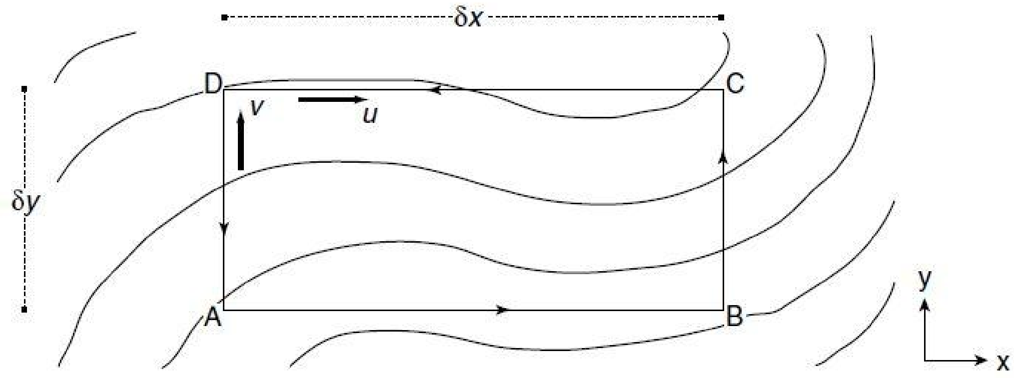


Components $\vec{\mathbf{V}}_c$ of a gradient flow $\vec{\mathbf{V}}_g$ that contribute to the circulation around the closed curve

On a Cartesian framework

$$\zeta \equiv \hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{V}} \right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\zeta \equiv \frac{\underset{A \to 0}{Lim} \left(\oint \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}} \right)}{A} \equiv \frac{\text{Circulation}}{\text{Area}}$$



Considering the circulation about a rectangular element of area $\delta x \delta y$ in the (x, y) plane.

Evaluating $\vec{\mathbf{V}} \cdot d\vec{\mathbf{l}}$ for each side of the triangle gives the circulation

$$C = \oint \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}} = \oint \left(u dx + v dy \right)$$

The circulation around an infinitesimal fluid element $ABCD$: $\delta C = C_{AB} + C_{BC} + C_{CD} + C_{DA}$

$$\delta C = u\delta x + \left(v + \frac{\partial v}{\partial x}\delta x \right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y \right)\delta x - v\delta y = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta x \delta y = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta A$$

$$\frac{\text{Circulation}}{\text{Area}} \equiv \frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \equiv \zeta$$

Vorticity and circulation in a rotating reference frame

Absolute vorticity $(\vec{\boldsymbol{\omega}}_a)$ = vorticity as viewed in an inertial reference frame.

Relative vorticity (ζ_{rel}) = vorticity as viewed in the rotating reference frame of the Earth.

Planetary vorticity $(\vec{\boldsymbol{\omega}}_{\text{pla}})$ = vorticity associated with the rotation of the Earth $(f = 2\Omega \sin\varphi)$.

$\Rightarrow$ In Meteorology, by the term circulation, we mean the circulation of velocity vector $\vec{\mathbf{V}}$

Vorticity vector:

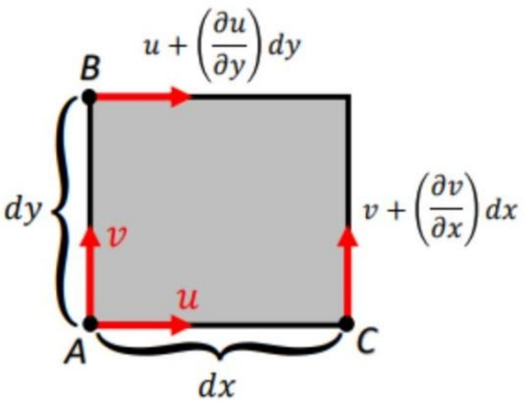
$$\vec{\boldsymbol{\omega}}_a \equiv \nabla \times \vec{\mathbf{V}}_{\text{Inertial}} = \vec{\zeta}_{\text{rel}} + 2\Omega \sin\varphi$$

Relative vorticity

$$\vec{\zeta}_{\text{rel}} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}} \text{ (Cartesian)}$$

$$\vec{\zeta}_{\text{rel}} = \left(\frac{1}{a}\frac{\partial v}{\partial \lambda} - \frac{1}{a\cos\varphi}\frac{\partial (u\cos\varphi)}{\partial y}\right)\hat{\mathbf{k}} \text{ (spherical)}$$

Planetary vorticity $(f) = 2\Omega \sin\varphi$



Cyclonic

$$\frac{\partial v}{\partial x} > 0, \frac{\partial u}{\partial y} < 0 \Rightarrow \zeta > 0$$

$$\Gamma \equiv \zeta \times \text{ Area } > 0$$

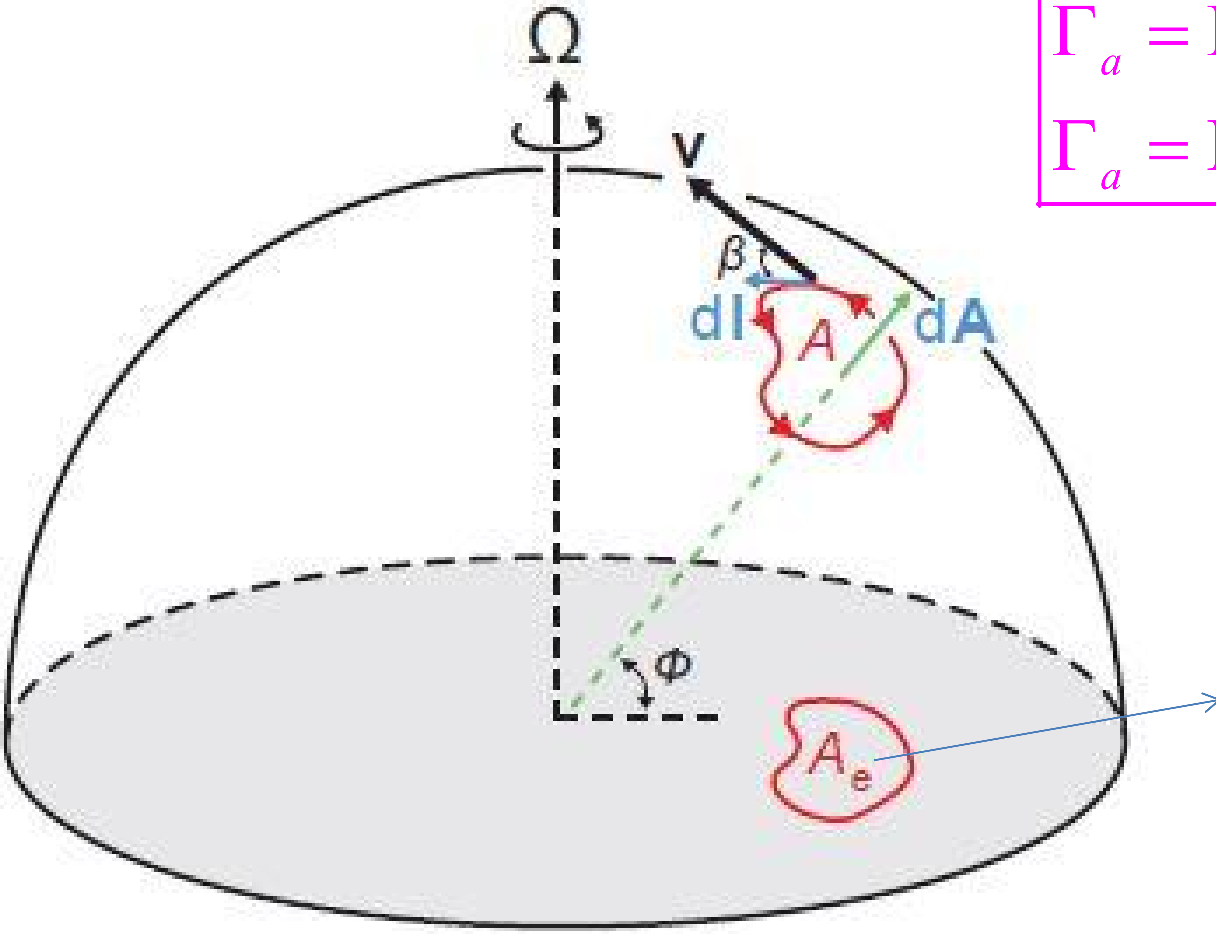
Anticyclonic

$$\frac{\partial v}{\partial x} < 0, \frac{\partial u}{\partial y} > 0 \Rightarrow \zeta < 0$$

$$\Gamma \equiv \zeta \times \text{ Area } < 0$$

Low pressure systems (cyclones): $\Gamma > 0$, $\zeta > 0$, Anticlockwise flow.

High pressure systems (anticyclones): $\Gamma < 0$, $\zeta < 0$, Clockwise flow.



$$\Gamma_a = \Gamma_r + 2\Omega A_e$$
$$\Gamma_a = \Gamma_r + (2\Omega \sin\varphi) A$$

Area projected on the equatorial plane

The absolute circulation is related to the relative circulation by $\Gamma_a = \Gamma_r + 2\Omega A_e$

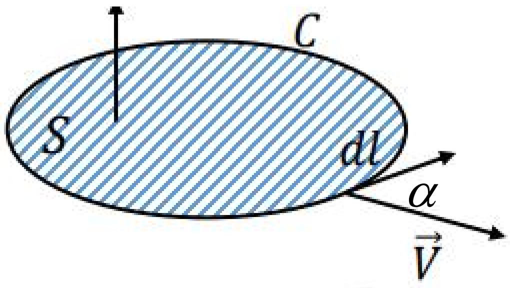
where A_e ($\equiv A \sin \varphi$) is the component of the area of the loop considered that is perpendicular to the rotation axis of the Earth

circulation ($2\Omega \sin\varphi A$) = planetary vorticity ($2\Omega \sin\varphi$) × area (A)

Vorticity and circulation

Circulation is a scalar measure of fluid rotation.

If we consider a closed path c around the fluid flow then circulation (Γ) is defined as the line integral of the tangential velocity around c



$$\Gamma \equiv \oint_c \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}} = \oint_c \left|\vec{\mathbf{V}}\right| \cos\alpha \; dl$$

By applying the Stoke's theorem, circulation can be related to the vorticity as;

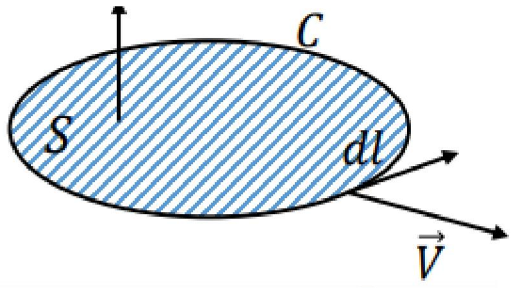
$$\Gamma = \int_c \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}} = \iint_S \left(\nabla \times \vec{\mathbf{V}}\right) \cdot \hat{\mathbf{n}} \; dS = \iint_S \vec{\boldsymbol{\omega}} \cdot \hat{\mathbf{n}} \; dS$$

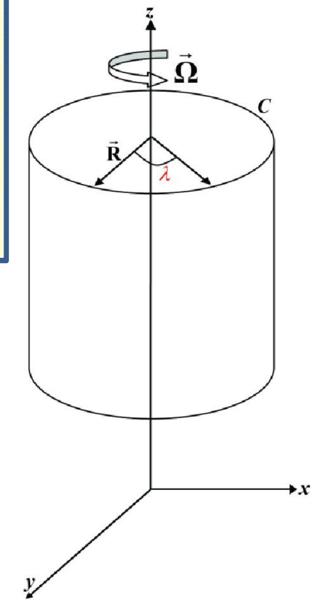
In conclusion, vorticity and circulation are two primary measures of rotation in fluid flow. Circulation is a macroscopic scalar measure of rotation for a given area of the fluid. However, vorticity being a vector quantity is a microscopic measure of rotation for any point in the flow.

Circulation and angular velocity

Considerations of angular momentum of fluid parcels is particularly important in understanding atmospheric dynamics

$\Rightarrow$ Circulation has the advantage over angular velocity that no assumption of a solid body is required and so it is suited to describing angular momentum ideas in a fluid.





Consider a circular ring of fluid of radius R in solid-body rotation at angular velocity $\vec{\mathbf{\Omega}}$ about the z-axis. The velocity field can be written as: $\vec{\mathbf{U}} = \vec{\mathbf{\Omega}} \times \vec{\mathbf{R}}$ where $\vec{\mathbf{R}}$ is the distance from the axis of rotation to the ring of fluid

$$C = \oint \vec{\mathbf{U}} \cdot d\vec{\mathbf{l}} = \int_0^{2\pi} \Omega R^2 d\lambda = 2\Omega\left(\pi R^2\right)$$

$$\frac{C}{\pi R^2} = 2\Omega = \boxed{\text{twice the angular speed of rotation of the ring}}$$

The fundamental definition of vorticity is (2Ω), that is, twice the local angular velocity

Vorticity however has nothing to do with a path, it is defined at a point and would indicate the rotation in the flow field at that point.

Unlike angular momentum or angular velocity, circulation can be computed without reference to an axis of rotation; it can thus be used to characterize fluid rotation in situations where "angular velocity" is not easily defined

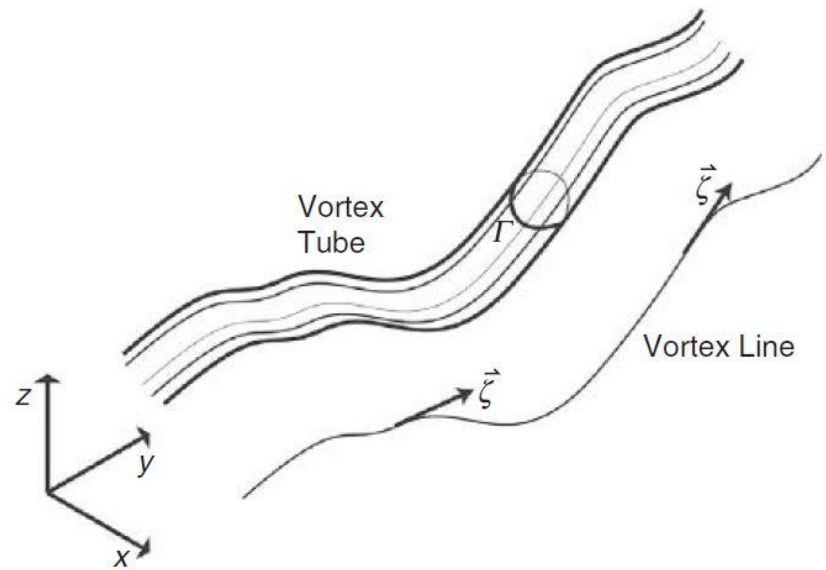
Circulation

Circulation is a scalar that measures the rotational property of a flow stemming from the notions of vortex lines and vortex tubes

$\Rightarrow$ A vortex line is defined to be a line to which the vorticity vector is tangential at every point of it (analogous to streamline)

$\Rightarrow$ The totality of all vortex lines passing through Γ would make up the surface of a tube is called vortex tube

$\Rightarrow$ The fluid inside such a tube is called a vortex filament



We now define a line integral of the velocity component tangential to this chosen closed curve (Γ). The sign convention is that this integration is performed in an anticlockwise direction

$$C = \oint_{\Gamma} \vec{\mathbf{V}} \cdot d\mathbf{l}$$

Exact differentials

Exact differentials: which are differentials whose integral around a close path is zero.

If a function f of two variables x and y has a differential that is written as

$df = Mdx + Ndy \rightarrow$ then it is an exact differential if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Leftrightarrow \boxed{\oint df \equiv 0}$

$\Rightarrow$ If a diffential is function of only a single variable and is of the form, $df = M(x)dx$, then it is an exact differential as long as M is integrable (f is differentiable).

Conservative vector fields $\equiv$ which are vector fields $\vec{\mathbf{g}}$ (gravity vector) that can be written in terms of the gradient of a scalar, such as "geopotential" $\boxed{\vec{\mathbf{g}} = \nabla\Phi}$

A conservative vector field has the property $\boxed{\oint \vec{\mathbf{g}} \cdot d\vec{\mathbf{l}} = \oint \nabla\Phi \cdot d\vec{\mathbf{l}} = \iint_A (\nabla \times \nabla\Phi) \cdot d\vec{\mathbf{A}} = 0}$

Remember the vector property : curl of gradient of a scalar function = 0 $\Leftrightarrow$ $\boxed{\nabla \times \nabla\Phi = 0}$

$C = \oint \vec{\mathbf{V}} \cdot d\vec{\mathbf{l}} = \oint (udx + vdy) \Rightarrow$ if $\vec{\mathbf{V}}$ is conservative, $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Leftrightarrow \boxed{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow \zeta \equiv 0}$

If $\nabla \times \vec{\mathbf{V}} = 0$ everywhere, then the field is called "irrotational", thus curl free field is conservative, A rotational vector is a vector field whose curl can never be zero.

Circulation theorems

Circulation theorems deal with the change or evolution in circulation and its cause(s).

For an arbitrary vector field ($\vec{\mathbf{B}}$), the circulation theorem states that the time rate of change of circulation of $\vec{\mathbf{B}}$ is equal to the circulation of the time rate of change of $\vec{\mathbf{B}}$

Circulation theorem: $\Leftrightarrow \dfrac{d}{dt}\oint \vec{\mathbf{B}} \cdot d\mathbf{l} = \oint \dfrac{d\vec{\mathbf{B}}}{dt} \cdot d\mathbf{l}$

Velocity vector on an inertial frame
$\vec{\mathbf{V}}_{\mathbf{a}} = \vec{\mathbf{V}}_{\mathrm{r}} + \vec{\mathbf{\Omega}} \times \vec{\mathbf{r}}$

$\Rightarrow$ Kelvin's circulation theorem, it is applied to the absolute velocity $\vec{\mathbf{V}}_{\mathbf{a}}$ of fluid motion

$$\frac{dC_a}{dt} = \frac{d}{dt}\oint \vec{\mathbf{V}}_{\mathbf{a}} \cdot d\mathbf{l} \equiv \oint \frac{d\vec{\mathbf{V}}_{\mathbf{a}}}{dt} \cdot d\mathbf{l} + \oint \vec{\mathbf{V}}_{\mathbf{a}} \cdot \frac{d}{dt}(d\mathbf{l}) \equiv \oint \frac{d\vec{\mathbf{V}}_{\mathbf{a}}}{dt} \cdot d\mathbf{l} + \oint d\left(\frac{\vec{\mathbf{V}}_{\mathbf{a}} \cdot \vec{\mathbf{V}}_{\mathbf{a}}}{2}\right)$$

$\Rightarrow$ as the line integral of an exact differential around a closed path = Zero

$$\frac{d_a C_a}{dt} \equiv \oint \frac{d_a \vec{\mathbf{V}}_{\mathbf{a}}}{dt} \cdot d\mathbf{l} \Leftrightarrow$$

In Meteorology, circulation theorem simply states that the acceleration of circulation is equal to the circulation of acceleration

Circulation theorems: Corollary

$$\frac{d_a C_a}{dt} \equiv \oint \frac{d_a \vec{\mathbf{V}}_{\mathbf{a}}}{dt} \cdot d\mathbf{l}$$

Equation for absolute motion is given by: $\frac{d_a \vec{\mathbf{V}}_a}{dt} = -\frac{1}{\rho}\nabla p + \vec{\mathbf{g}}$

$$\oint \frac{d_a \vec{\mathbf{V}}_{\mathbf{a}}}{dt} \cdot d\mathbf{l} = -\oint \frac{1}{\rho}\nabla p \cdot d\mathbf{l} + \cancel{\oint \vec{\mathbf{g}} \cdot d\mathbf{l}} = \iint_S \left(-\frac{1}{\rho}\nabla p\right) \cdot \hat{\mathbf{n}}\, dS = \iint_S \left(\frac{\nabla\rho \times \nabla p}{\rho^2}\right) \cdot \hat{\mathbf{n}}\, dS$$

$\Rightarrow$ $\vec{\mathbf{g}}$ is a conservative force field. It is also known that work done by a conservative force field around a closed path is zero $\oint \vec{\mathbf{g}} \cdot d\mathbf{l} = \oint \nabla\Phi \cdot d\mathbf{l} = \oint d\Phi \equiv 0$

Therefore, for a frictionless flow, $\frac{d_a C_a}{dt} = \iint_S \left(\frac{\nabla\rho \times \nabla p}{\rho^2}\right) \cdot \hat{\mathbf{n}}\, dS$

gravity force would not be involved in the generation of rotating motion

$\Rightarrow$ in a barotropic atmosphere the density, $\rho = \rho(p) \Leftrightarrow \nabla\rho \times \nabla p = 0$

Therefore, for a frictionless barotropic flow, $\frac{d_a C_a}{dt} = 0 \Leftrightarrow$ Circulation is conserved

This is a direct corollary to the Kelvin's theorem. Hence from Kelvin's circulation theorem it may be stated that for frictionless flow change in absolute circulation is solely due to the baroclinicity of the atmosphere.

$\frac{d_a C_a}{dt} = \oint \frac{dp}{\rho} \neq 0 \Leftrightarrow$ Baroclinic fluid $\nabla\rho \times \nabla p \neq 0$ $\Leftrightarrow$ integrated effect of pressure-gradient force can change the absolute circulation $\Rightarrow$ Bjerknes circulation theorem

Baroclinicity

$$\text{Baroclinicity term} = -\oint \frac{1}{\rho}\nabla p \cdot d\mathbf{l} = -\oint \alpha \nabla p \cdot d\mathbf{l} \qquad \boxed{\alpha \equiv \frac{1}{\rho}}$$

$$-\oint \alpha \nabla p \cdot d\mathbf{l} = -\int_A \nabla \times \left(\alpha \nabla p\right) \cdot d\mathbf{A}$$

$$-\iint \left(\nabla \times \alpha \nabla p\right) \cdot d\vec{\mathbf{l}} = -\cancel{\int_A \alpha\left(\nabla \times \nabla p\right) \cdot d\mathbf{A}} - \int_A \left(\nabla \alpha \times \nabla p\right) \cdot d\mathbf{A}$$

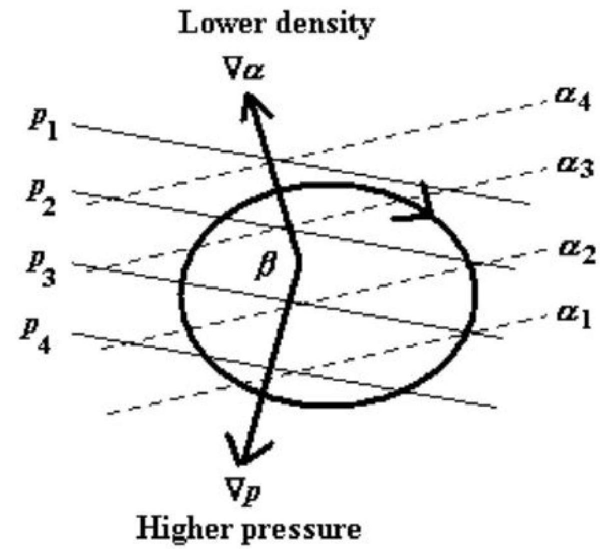
$$\Rightarrow \boxed{\begin{aligned} &\frac{dC_a}{dt} = -\int_A \left(\nabla \alpha \times \nabla p\right) \cdot d\mathbf{A} = -\int_A \left|\nabla \alpha\right|\left|\nabla p\right| \sin\beta \; dA \\ &\beta \equiv \begin{cases} 0 - \pi \Rightarrow \text{ clockwise} \\ \pi < \beta < 2\pi \Rightarrow \text{ anticlockwise} \end{cases} \end{aligned}}$$

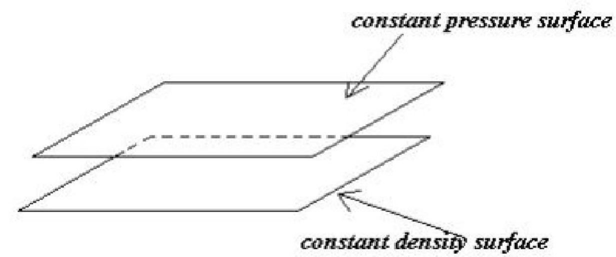
$\beta \equiv$ Angle between the gradients of α and p

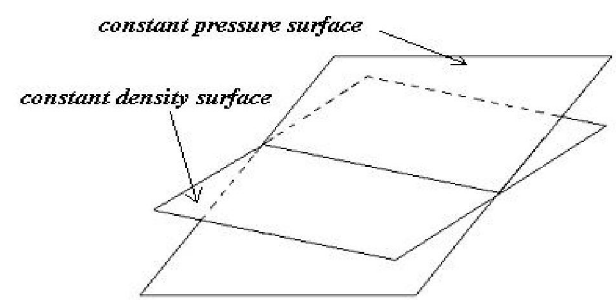
$\Rightarrow$ In the given example $\dfrac{dC_a}{dt} < 0$

For the atmosphere, which is an ideal gas, the solenoidal term can be written in terms of the temperature and pressure gradients as

$$\boxed{\frac{dC_a}{dt} = -R\iint_A \nabla T \times \nabla\left(\ln p\right) \cdot d\mathbf{A}}$$







Bjerknes circulation theorem - examples

$$\frac{dC_{abs}}{dt} \neq 0 \text{ in a baroclinic fluid}$$

$$\frac{dC_{abs}}{dt} = \frac{d}{dt}\left(C_{rel} + 2\Omega A_e\right) = -\oint \frac{dp}{\rho}$$

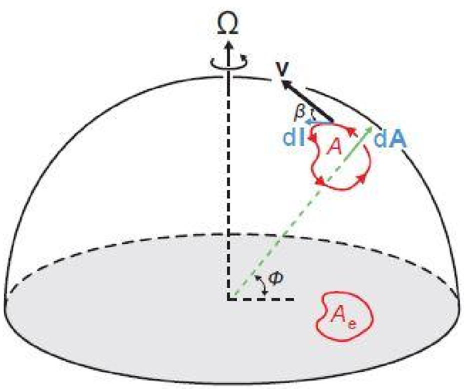
$$\frac{dC_{abs}}{dt} = -\oint RTd\ln p$$

(along a vertical plane) f is negligible

where C is circulation, A_e is the area of the integral circuit projected onto the equatorial plane. The term on the right-hand side represents solenoidal or baroclinic generation of circulation and $2\Omega A_e$ term represents the change in circulation due to rotation of the Earth.

$\Rightarrow$ In a baroclinic fluid, circulation may be generated by the pressure-density solenoid term

Below we present an example of how Bjerknes' circulation theorem can be used to predict the circulation induced by a horizontal temperature gradient such as might be found along a mesoscale air mass boundary.i.e., we apply the circulation theorem by integrating around a circuit in a vertical plane



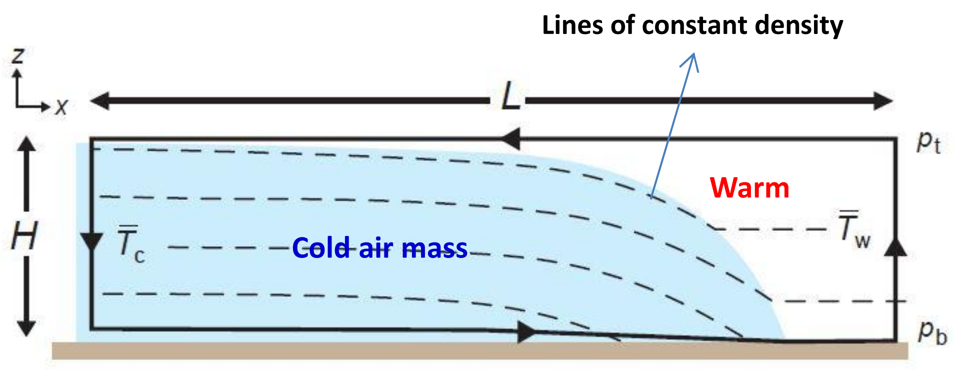
$$\frac{dC}{dt} = -\oint RTd\ln p$$

$$\frac{dC}{dt} = R\ln\left(\frac{p_b}{p_t}\right)\left(\overline{T}_w - \overline{T}_c\right) > 0$$

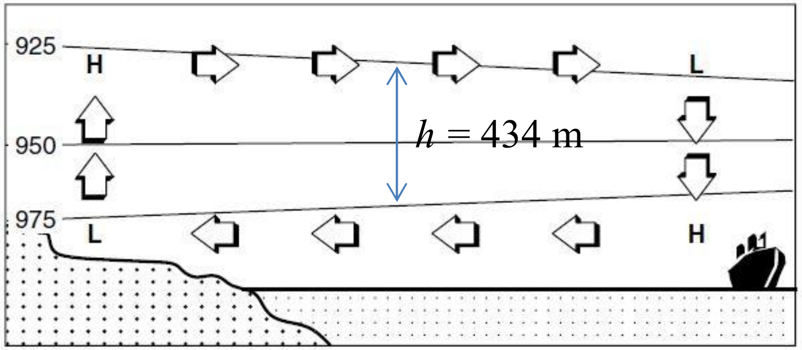
$p_t = 900$ hPa, $p_b = 1000$ hPa

$\overline{T}_w = 300$ K, $\overline{T}_c = 290$ K

$\frac{dC}{dt} = 302\ \text{m}^2\text{s}^{-2} \Leftrightarrow$ an increase of C of $1.09\times10^6\ \text{m}^2\text{s}^{-1}$ in 1 hour



Land and sea breeze
See Holton's book for more details



$$\frac{dC_a}{dt} = \oint \frac{dp}{\rho} = -\oint RT d\ln p$$

$$\frac{dC_a}{dt} = R\ln\left(\frac{p_{975}}{p_{925}}\right)\left(\bar{T}_{LAND} - \bar{T}_{OCEAN}\right) = -302\ \mathrm{m^2s^{-2}}$$

temperature difference (20^oC) is inducing a clockwise circulation

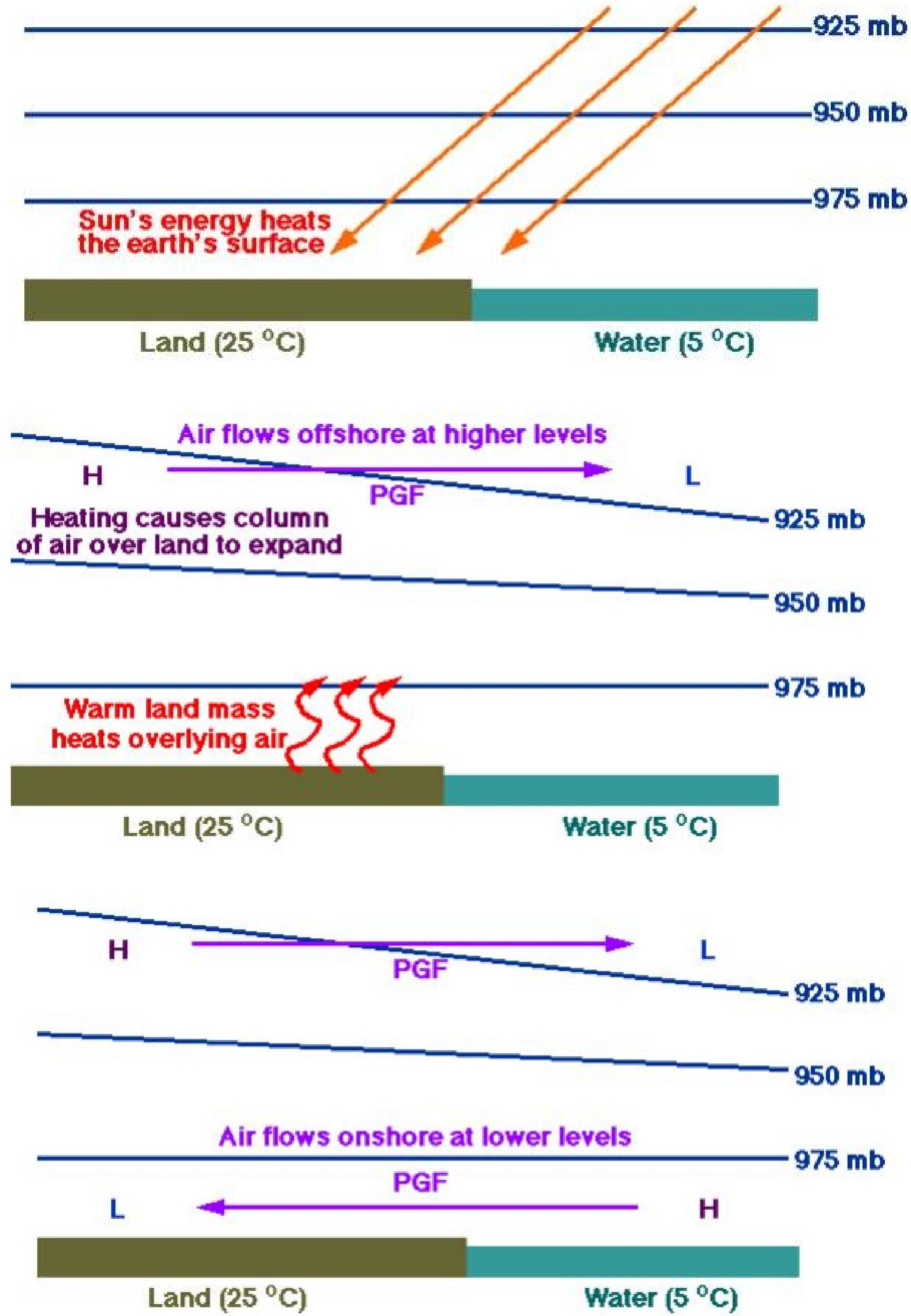
To estimate wind acceleration,

$$\frac{dC}{dt} = \oint \frac{d\vec{\mathbf{V}}}{dt}\cdot d\mathbf{l} \approx \frac{d\vec{\mathbf{V}}}{dt}\cdot\oint d\mathbf{l} \Rightarrow \boxed{\frac{du}{dt} \approx \frac{dC/dt}{2(L+h)}}$$

$$\frac{du}{dt} = \frac{R\ln\left(p_{975}/p_{925}\right)}{2(L+h)}\left(\bar{T}_L - \bar{T}_O\right) = -7.4\times10^{-3}\ \mathrm{m^2s^{-2}}$$

$L = 20$ km; $h = 434$ m (using hypsometric equation)

$\Leftrightarrow$ which could produce a wind speed of 25 m $\mathrm{s^{-1}}$ in 1 hour in the absence of friction



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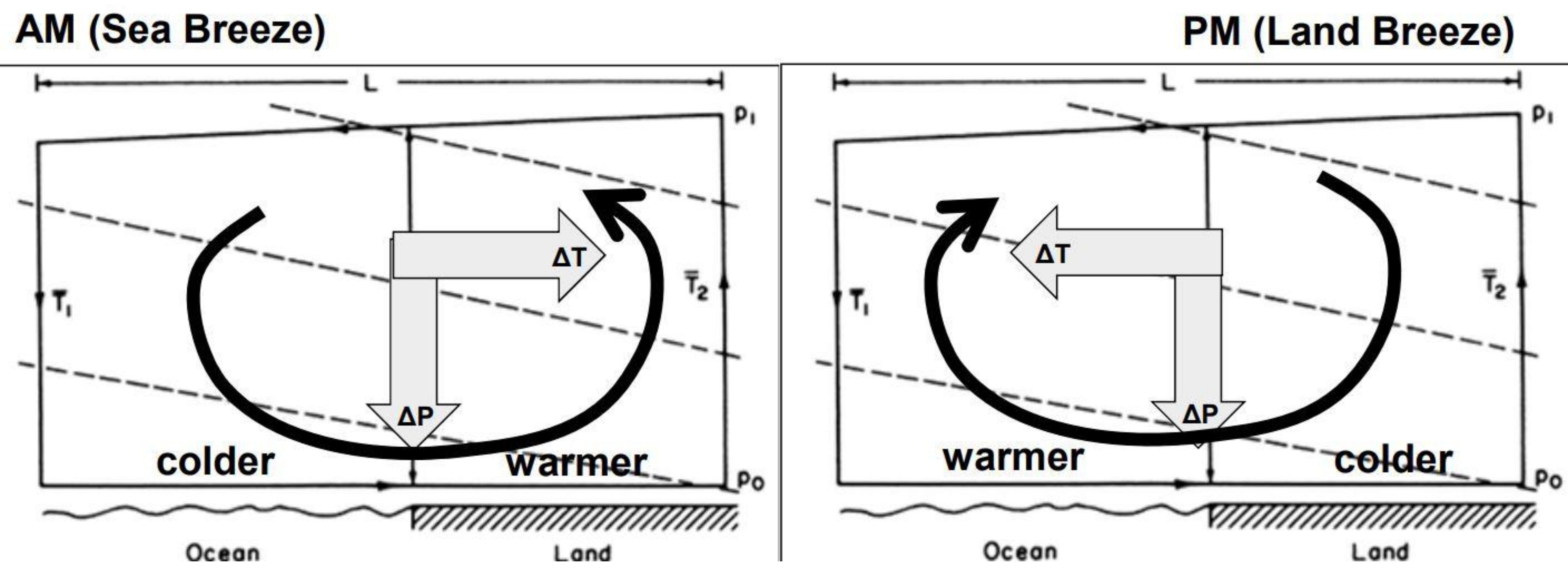
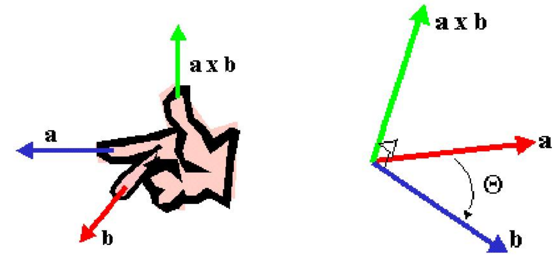
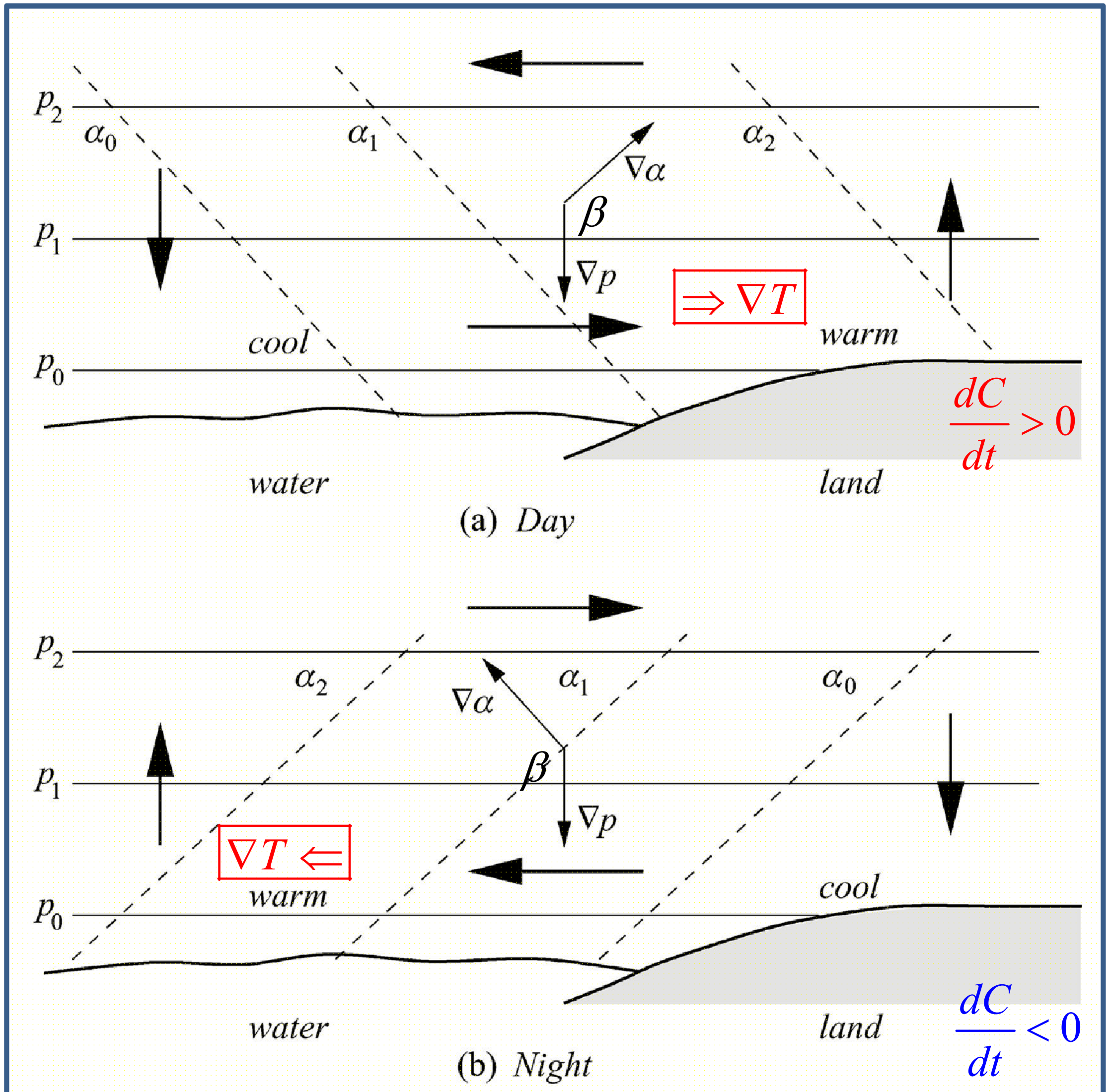
$$\frac{dC_a}{dt} = -R\iint_A \nabla T \times \nabla(\ln p) \cdot d\mathbf{A}$$



Solenoidal or baroclinicity





$$\frac{dC}{dt} = -\iint_A (\nabla\alpha \times \nabla p)\cdot d\vec{\mathbf{A}}; \quad \alpha = 1/\rho$$

$$\frac{dC}{dt} = -R\iint_A \nabla T \times \nabla(\ln p)\cdot d\vec{\mathbf{A}}$$

$$\frac{dC}{dt} = -\int_A |\nabla\alpha||\nabla p|\sin\beta \; dA$$

$$\frac{dC}{dt} = -\int_A |\nabla T||\nabla \ln p|\sin\beta \; dA$$

If the atmosphere is barotropic

$$\hat{\mathbf{k}}\cdot(\nabla p \times \nabla\alpha) = 0$$

Pressure and density surfaces lie on each other (barotropic atmosphere)

If the atmosphere is baroclinic

$$\hat{\mathbf{k}}\cdot(\nabla p \times \nabla\alpha) > 0 \;\Rightarrow\; \text{cyclonic}$$

$$\hat{\mathbf{k}}\cdot(\nabla p \times \nabla\alpha) < 0 \;\Rightarrow\; \text{anticyclonic}$$

In terms of temperature gradients,

Low level winds flow in the direction of $\nabla T > 0$

Vorticity and circulation in a rotating reference frame

Kelvin's circulation theorem which states that if the fluid is barotropic on the material curve C and the frictional force on C is zero then absolute circulation is conserved following the motion of the fluid

In a rotating fluid, the velocity vector (on an inertial frame) $\vec{\mathbf{V}}_{\text{abs}} = \vec{\mathbf{V}}_{\text{rel}} + \left(\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{r}}\right)_{\text{frame}}$

so that the vorticity associated with the velocity in an inertial frame is related to the velocity in a rotating frame by $\vec{\boldsymbol{\omega}}_{\text{abs}} = \vec{\boldsymbol{\omega}}_{\text{rel}} + \nabla \times \left(\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{r}}\right)_{\text{frame}} \Rightarrow \vec{\boldsymbol{\omega}}_{\text{abs}} = \vec{\boldsymbol{\omega}}_{\text{rel}} + 2\vec{\boldsymbol{\Omega}}$ where $\vec{\boldsymbol{\omega}} = \nabla \times \vec{\mathbf{V}}$

$\Leftrightarrow$ so the vorticity in the inertial frame is equal to the vorticity seen in the rotating frame called the relative vorticity plus the vorticity of the velocity due to the frame's rotation which is just twice the rotation rate of the frame

To examine the form of Kelvin's theorem for a rotating frame

$$C_{abs} = C + \oint_c \left(\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{r}}\right) \cdot d\vec{\mathbf{r}}$$

$$C_{abs} = C + \int_A 2\vec{\boldsymbol{\Omega}} \cdot \hat{\mathbf{n}}\, dA = C + 2\Omega A_e$$

The circulation observed in a rotating frame

$$\frac{dC_{abs}}{dt} = \frac{dC}{dt} + 2\Omega\frac{dA_e}{dt} = \int_A \frac{\nabla\rho \times \nabla p}{\rho^2} \cdot \hat{\mathbf{n}}\, dA$$

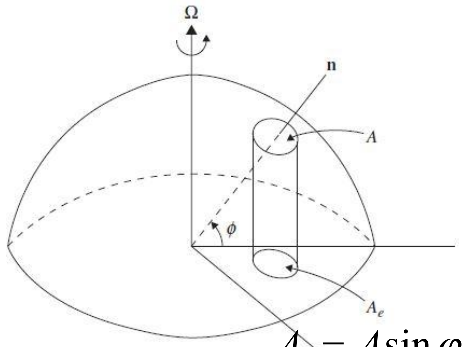
Consider the situation where viscosity can be neglected and where the fluid is barotropic $\left(\nabla\rho \times \nabla p = 0\right)$.

$$\frac{dC}{dt} + 2\Omega\frac{dA_e}{dt} = 0 \Rightarrow$$

which just tells us that the absolute circulation is conserved

$\Leftrightarrow$ In other words there will be a trade off between the relative and planetary vorticities

$$\Rightarrow C_2 - C_1 = -2\Omega\left(A_2 \sin\varphi_2 - A_1 \sin\varphi_1\right)$$



$A_e = A\sin\varphi$

Vorticity and circulation in a rotating reference frame

To examine the form of Kelvin's theorem for a rotating frame

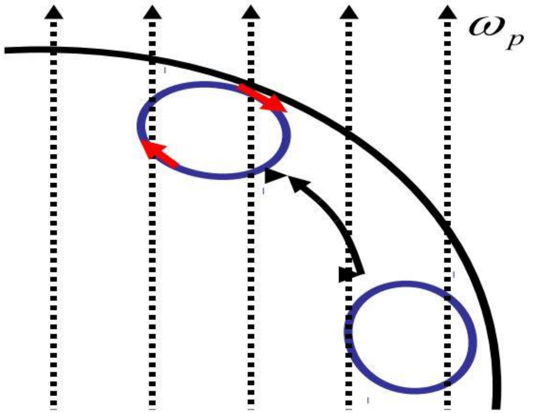
$$C_{abs} = C + \oint_c \left(\vec{\mathbf{\Omega}} \times \vec{\mathbf{r}}\right)\cdot d\vec{\mathbf{r}}$$

$$C_{abs} = C + \int_A 2\vec{\mathbf{\Omega}}\cdot\hat{\mathbf{n}}\; dA = C + 2\Omega A_e$$

The circulation observed in a rotating frame

$$\frac{dC_{abs}}{dt} = \frac{dC}{dt} + 2\Omega\frac{dA_e}{dt} + \int_A \frac{\nabla\rho\times\nabla p}{\rho^2}\cdot\hat{\mathbf{n}}\; dA$$

Consider the barotropic situation where viscosity can be neglected



$$\frac{dC}{dt} + 2\Omega\frac{dA_e}{dt} = 0 \Rightarrow$$

which just tells us that the absolute circulation is conserved $\Leftrightarrow$ In other words there will be a trade off between the relative and planetary vorticities

$$\Rightarrow C_2 - C_1 = -2\Omega\left(A_2\sin\varphi_2 - A_1\sin\varphi_1\right)$$

$\Rightarrow$ A material curve that is shifted to higher latitude. As the curve moves to higher latitude the area normal to the Earth's rotation axis will increase and so the circulation associated with the planetary vorticity increases ($2\Omega A_e$)

$\Rightarrow$ In order to conserve the absolute circulation the relative circulation must decrease (an anticyclonic circulation is induced)

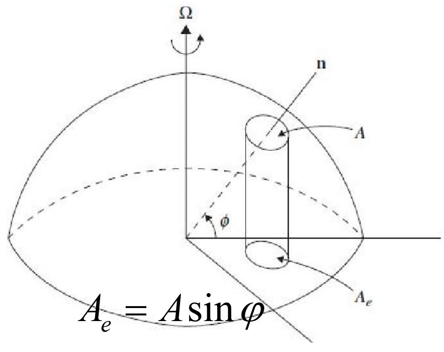
An air parcel at 30°N moves northward conserving absolute vorticity. If its initial relative vorticity is $5\times10^{-5}\ \mathrm{s}^{-1}$, what is its relative vorticity upon reaching 90°N?

$$(\zeta + f)_{30} = (\zeta + f)_{90} \Rightarrow (5\times10^{-5} + \Omega)_{30} = (\zeta_2 + 2\Omega)_{90}$$

$$\zeta_2 \approx -2.3\times10^{-5}\mathrm{s}^{-1}$$

Implication : conservation of absolute vorticity (CAV)

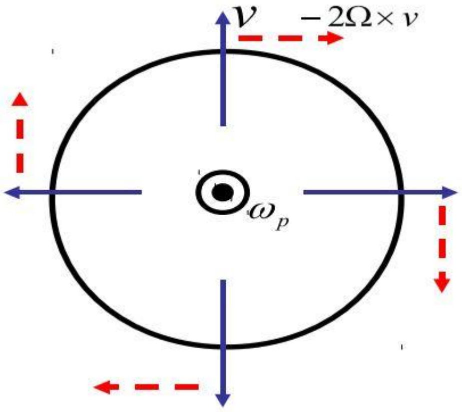
a negative absolute circulation in the Northern Hemisphere can develop only if a closed chain of fluid particles is advected poleward $\Leftrightarrow$ Rossby wave $\rightarrow$ CAV trajectory



Vorticity and circulation in a rotating reference frame

The rate of change of the relative circulation

$$\frac{d}{dt}\oint_c \vec{\mathbf{V}} \cdot d\mathbf{l} = \frac{dC}{dt} = \oint_c \left(-2\vec{\mathbf{\Omega}} \times \vec{\mathbf{V}}\right) \cdot d\mathbf{l} - \oint_c \frac{\nabla p}{\rho} \cdot d\mathbf{l} + \oint_c \vec{\mathbf{F}}_{\text{friction}} \cdot d\mathbf{l}$$



where $\vec{\mathbf{F}}_{\text{friction}}$ is the frictional force per unit mass. There are therefore three terms that can act to alter the circulation.

$$\oint_c \left(-2\vec{\mathbf{\Omega}} \times \vec{\mathbf{V}}\right) \cdot d\mathbf{l}$$

Consider the circulation around the curve C in a divergent flow. It is clear that the coriolis force acting on the flow field acts to induce a circulation around the curve C

Consider friction to be a linear drag on the velocity with some timescale τ

$$\frac{dC}{dt} = \oint_c \vec{\mathbf{F}}_{\text{friction}} \cdot d\mathbf{l} = -\frac{1}{\tau}\oint_c \vec{\mathbf{V}} \cdot d\mathbf{l} = -\frac{C}{\tau} \Rightarrow$$

friction acts to spin-down the circulation

spin-down = decay

Helmholtz theorem

Theorem: Every vector field $\vec{\mathbf{V}}$ whose divergence and rotation possess potentials can be written as the sum of a divergence-free vector field plus another vector field that is irrotational.

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}_{\psi} + \vec{\mathbf{V}}_{\chi} = \hat{\mathbf{k}} \times \vec{\nabla}\psi + \vec{\nabla}\chi \Leftrightarrow$$

non-divergent (divergence-free; $\vec{\mathbf{V}}_{\psi}$)

divergent (irrotational/curl-free; $\vec{\mathbf{V}}_{\chi}$)

where ψ is the stream function and χ is the velocity potential. The relative vorticity (ς) and divergence (δ) can be expressed as follows:

$$\varsigma = \hat{\mathbf{k}} \cdot \nabla \times \vec{\mathbf{V}}_{\psi} = \frac{1}{a\cos\varphi}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(u\cos\varphi)\right] = \nabla^2\psi$$

$$\delta = \nabla \cdot \vec{\mathbf{V}}_{\chi} = \frac{1}{a\cos\varphi}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\varphi}(v\cos\varphi)\right] = \nabla^2\chi$$

$$\vec{\mathbf{V}}_{\psi} = -\frac{1}{a}\frac{\partial\psi}{\partial\varphi}\hat{\mathbf{i}} + \frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}\hat{\mathbf{j}}; \qquad \vec{\mathbf{V}}_{\chi} = \frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}\hat{\mathbf{i}} + \frac{1}{a}\frac{\partial\psi}{\partial\varphi}\hat{\mathbf{j}}$$

Typically, $\left|\vec{\mathbf{V}}_{\psi}\right| >> \left|\vec{\mathbf{V}}_{\chi}\right|$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$u_{\psi} = -\frac{\partial\psi}{\partial y}; \; v_{\psi} = \frac{\partial\psi}{\partial x}$$

$$u_{\chi} = \frac{\partial\chi}{\partial x}; \; v_{\chi} = \frac{\partial\chi}{\partial y}$$

Helmholtz theorem

$$\vec{\mathbf{V}} = \vec{\mathbf{V}}_\psi + \vec{\mathbf{V}}_\chi = \hat{\mathbf{k}} \times \vec{\nabla}\psi + \vec{\nabla}\chi$$

Irrotational vector $\vec{\mathbf{V}}_\chi$

In vector calculus an irrotational vector field is a vector $\vec{\mathbf{V}}_\chi$ with curl zero at all points in the field $\nabla \times \vec{\mathbf{V}}_\chi = 0 \Leftrightarrow$ Any motion in which the curl of the velocity vector is zero is said to be "irrotational" $\rightarrow$ potential flow

In a vector calculus a solenoidal vector field is a vector field $\vec{\mathbf{V}}$ with zero divergence at all points in the field $\nabla \cdot \vec{\mathbf{V}}_\psi = 0 \Leftrightarrow$ Any motion in which the divergence of the velocity vector is zero is said to be "solenoidal" $\rightarrow$ can produce vorticity

$$\nabla \times \vec{\mathbf{V}} = \nabla \times \vec{\mathbf{V}}_\psi + \nabla \times \vec{\mathbf{V}}_\chi$$

$$\nabla \times \vec{\mathbf{V}} = \nabla \times \left(\hat{\mathbf{k}} \times \vec{\nabla}\psi\right) + \cancel{\nabla \times \vec{\nabla}\chi}$$

$\nabla \times \vec{\mathbf{V}}_\psi$ leads to $\nabla^2\psi \equiv \zeta$

rotational part of $\vec{\mathbf{V}} \Rightarrow$ vorticity

(Measure of the non-divergent part of the wind)

$$\nabla \cdot \vec{\mathbf{V}} = \nabla \cdot \vec{\mathbf{V}}_\psi + \nabla \cdot \vec{\mathbf{V}}_\chi$$

$$\nabla \cdot \vec{\mathbf{V}} = \cancel{\nabla \cdot \left(\hat{\mathbf{k}} \times \vec{\nabla}\psi\right)} + \nabla \cdot \nabla\chi$$

$\nabla \cdot \vec{\mathbf{V}}_\chi$ leads to $\nabla^2\chi \equiv \delta$

irrotational part of $\vec{\mathbf{V}} \Rightarrow$ divergence

(measure of the divergent part of the wind)

Stream function and velocity potential

Helmholtz decomposition:

The horizontal velocity field $\vec{\mathbf{V}}_H = \vec{\mathbf{V}}_\psi + \vec{\mathbf{V}}_\chi$

In terms of scalar functions, we can write $\vec{\mathbf{V}}_H = \hat{\mathbf{k}} \times \nabla\psi + \nabla\chi$

On a streamline, $\psi(x,y)$ is constant, so that the total differential $d\psi$=0

$$d\psi = \nabla\psi \cdot d\hat{\mathbf{s}} = 0$$

$$d\psi = \left(\hat{\mathbf{i}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\psi}{\partial y}\right)\cdot\left(\hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy\right) = 0$$

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \qquad (1)$$

Rotational part of the winds: $\vec{\mathbf{V}}_\psi = u_\psi\hat{\mathbf{i}} + v_\psi\hat{\mathbf{j}}$

$$\vec{\mathbf{V}}_\psi = \hat{\mathbf{k}} \times \nabla\psi = \left(-\frac{\partial\psi}{\partial y}\hat{\mathbf{i}} + \frac{\partial\psi}{\partial x}\hat{\mathbf{j}}\right)$$

$$\Rightarrow \quad u = -\frac{\partial\psi}{\partial y}; \quad v = \frac{\partial\psi}{\partial x} \qquad (2)$$

Substituting (2) in (1), $\quad vdx - udy = 0 \quad \Rightarrow \quad \boxed{\left(\frac{dy}{dx}\right)_\psi = \frac{v}{u}}$

This is the equation of the streamline.

On a line of constant velocity potential $\chi(x,y)$ is constant, so that $d\chi$=0

$$d\chi = \nabla\chi \cdot d\hat{\mathbf{s}} = 0$$

$$d\chi = \frac{\partial\chi}{\partial x}dx + \frac{\partial\chi}{\partial y}dy = 0 \qquad (3)$$

Irrotational part of the winds: $\vec{\mathbf{V}}_\chi = u_\chi\hat{\mathbf{i}} + v_\chi\hat{\mathbf{j}}$

$$\vec{\mathbf{V}}_\chi = \nabla\chi = \left(\frac{\partial\chi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\chi}{\partial y}\hat{\mathbf{j}}\right)$$

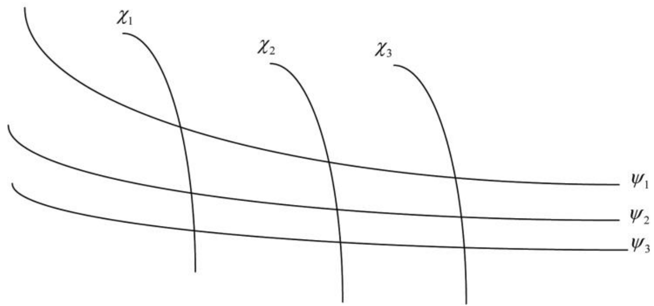
$$\Rightarrow \quad u = \frac{\partial\chi}{\partial x}; \quad v = \frac{\partial\chi}{\partial y} \qquad (4)$$

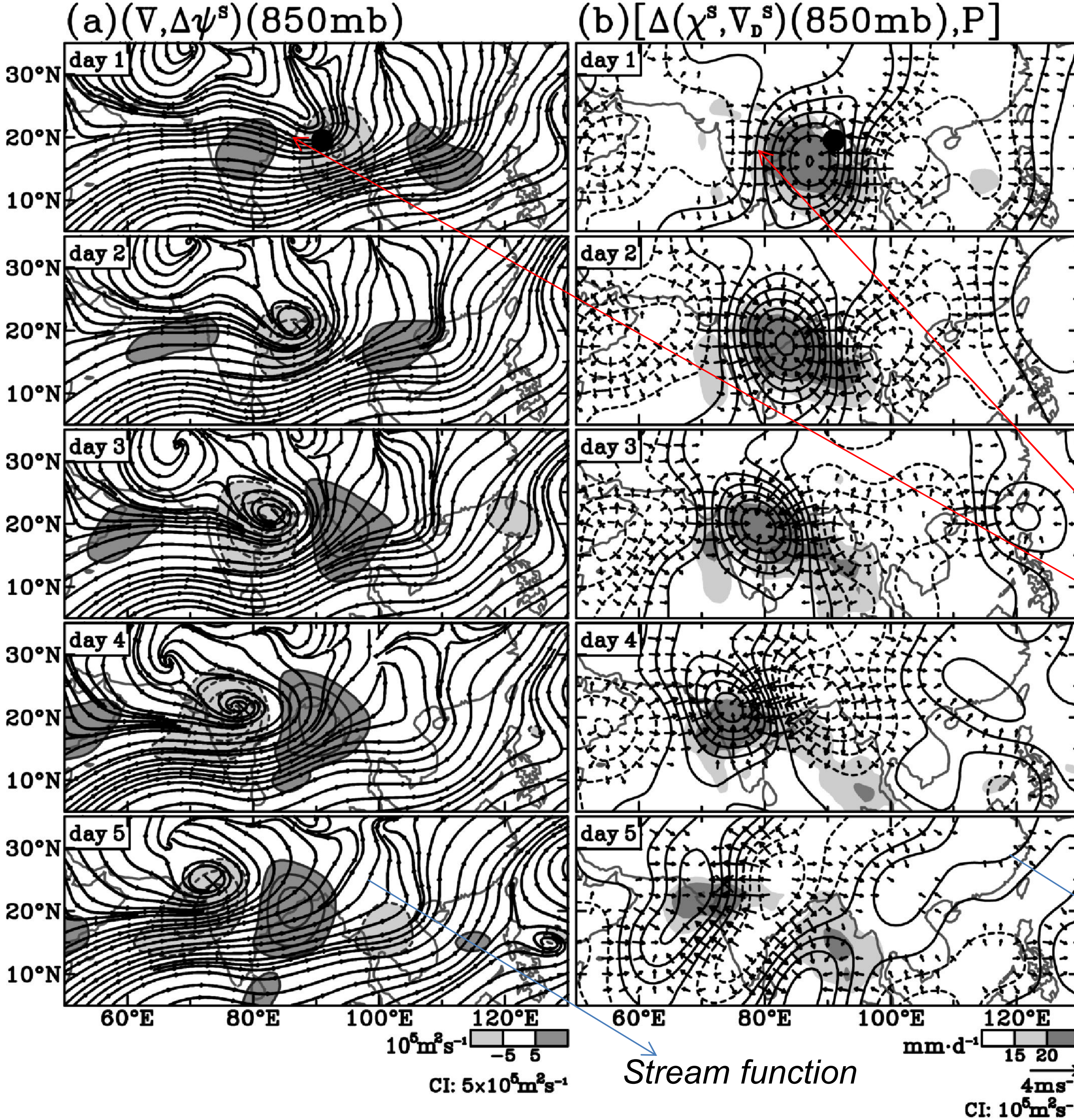
Substituting (4) in (3), $\quad udx + vdy = 0 \quad \Rightarrow \quad \boxed{\left(\frac{dy}{dx}\right)_\chi = -\frac{u}{v}}$

This is the equation of the velocity potential.

Since the slopes of a streamline and of a line of constant velocity potential are negative reciprocals of one another, streamlines and velocity potential lines must be mutually perpendicular.

Lines of constant ψ (streamlines) are perpendicular to lines of constant χ (velocity potential lines).

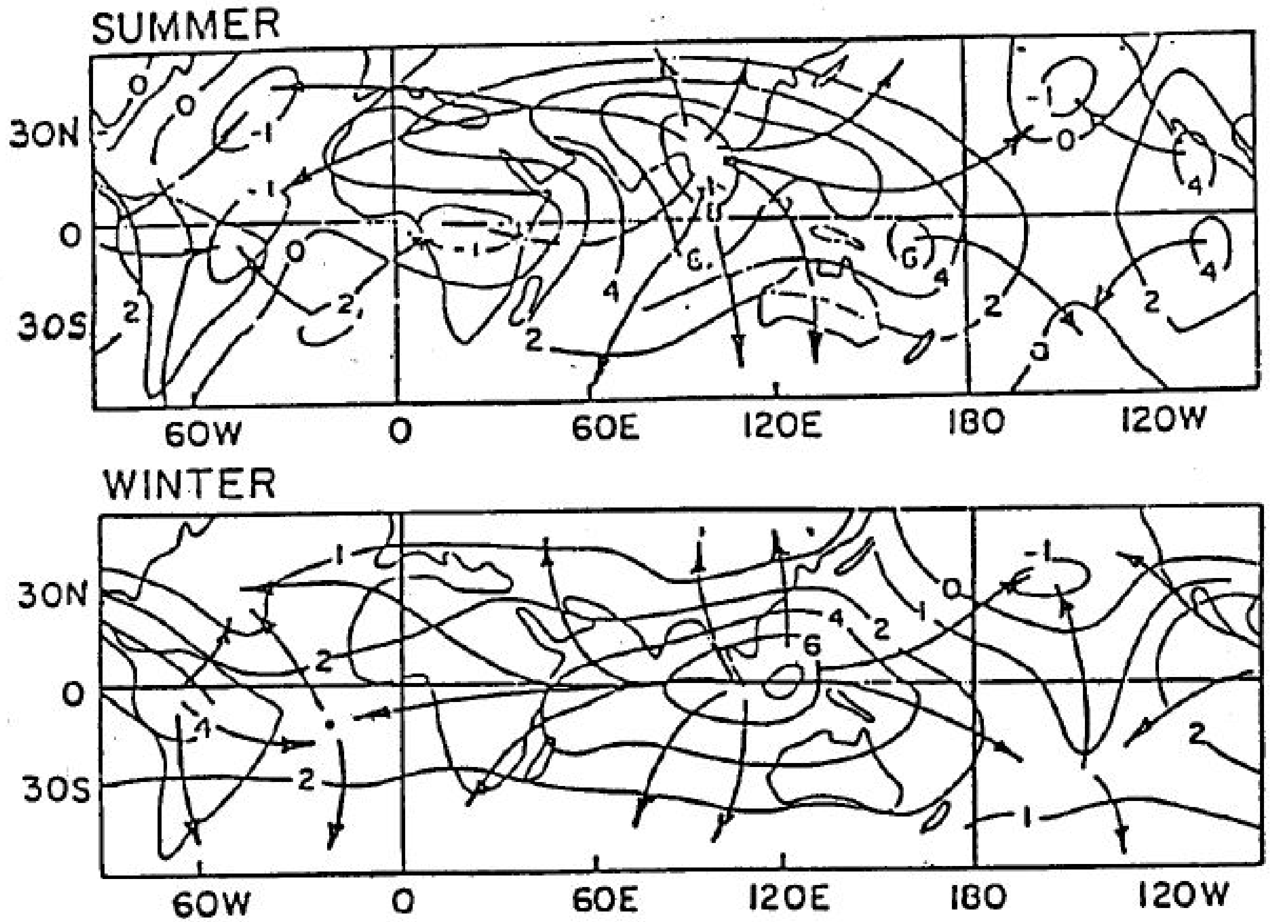




Composite monsoon depression depicted by the 850-mb streamline field superimposed with stream function departures (shaded areas) from their summer (JJA) mean values in the short-wave regime, $\Delta\psi^s$(850 mb), (b) composite 850-mb velocity potential and divergent wind departures in the short-wave regime superimposed with precipitation, [$\Delta(\chi s, \mathbf{V_D})$ (850 mb), P]

streamlines and velocity potential lines are mutually perpendicular to each other.

$\nabla\chi$ is irrotational, but has divergence $\nabla^2\chi$. Because of this last property, examination of the velocity potential is especially useful as a diagnostic tool for isolating the divergent circulation



Distribution of the upper tropospheric (200 hPa) mean seasonal velocity potential (solid lines) and arrows indicating the divergent part of the mean seasonal wind which is proportional to $\nabla^2\chi$. (Adapted from Krishnamurti et al. 1973).

In x - y system $\rightarrow$ Laplacian operator $\rightarrow$ "curvature of function or field"

$$\hat{k}\cdot\left(\nabla\times\vec{\mathbf{V}}\right)\Rightarrow\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\nabla^2\psi=\frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{\Delta x^2}+\frac{\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1}}{\Delta y^2}$$

$$\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}}=\delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\nabla^2\chi=\frac{\chi_{i+1,j}-2\chi_{i,j}+\chi_{i-1,j}}{\Delta x^2}+\frac{\chi_{i,j+1}-2\chi_{i,j}+\chi_{i,j-1}}{\Delta y^2}$$

$\Leftrightarrow$ It tells you how much the value of the field differs from its average value taken over the surrounding points.

What is the physical significance of the Laplacian?

In 1-D $\Rightarrow\nabla^2\psi$ reduces to $\dfrac{\partial^2\psi}{\partial x^2}$

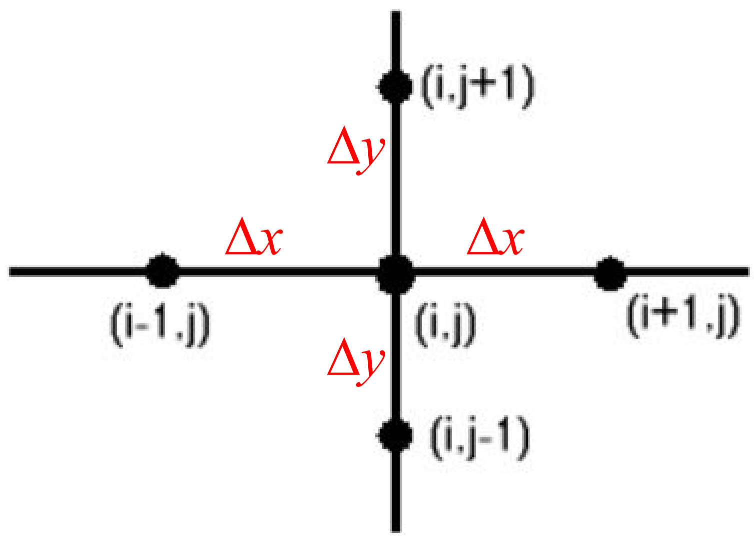
If $\dfrac{\partial^2\psi}{\partial x^2}>0\rightarrow\ \psi(x)$ is concave (convex if $\dfrac{\partial^2\psi}{\partial x^2}<0$)

$\dfrac{\partial^2\psi}{\partial x^2}>0\Rightarrow$ ψ is less than the average of ψ in its surroundings

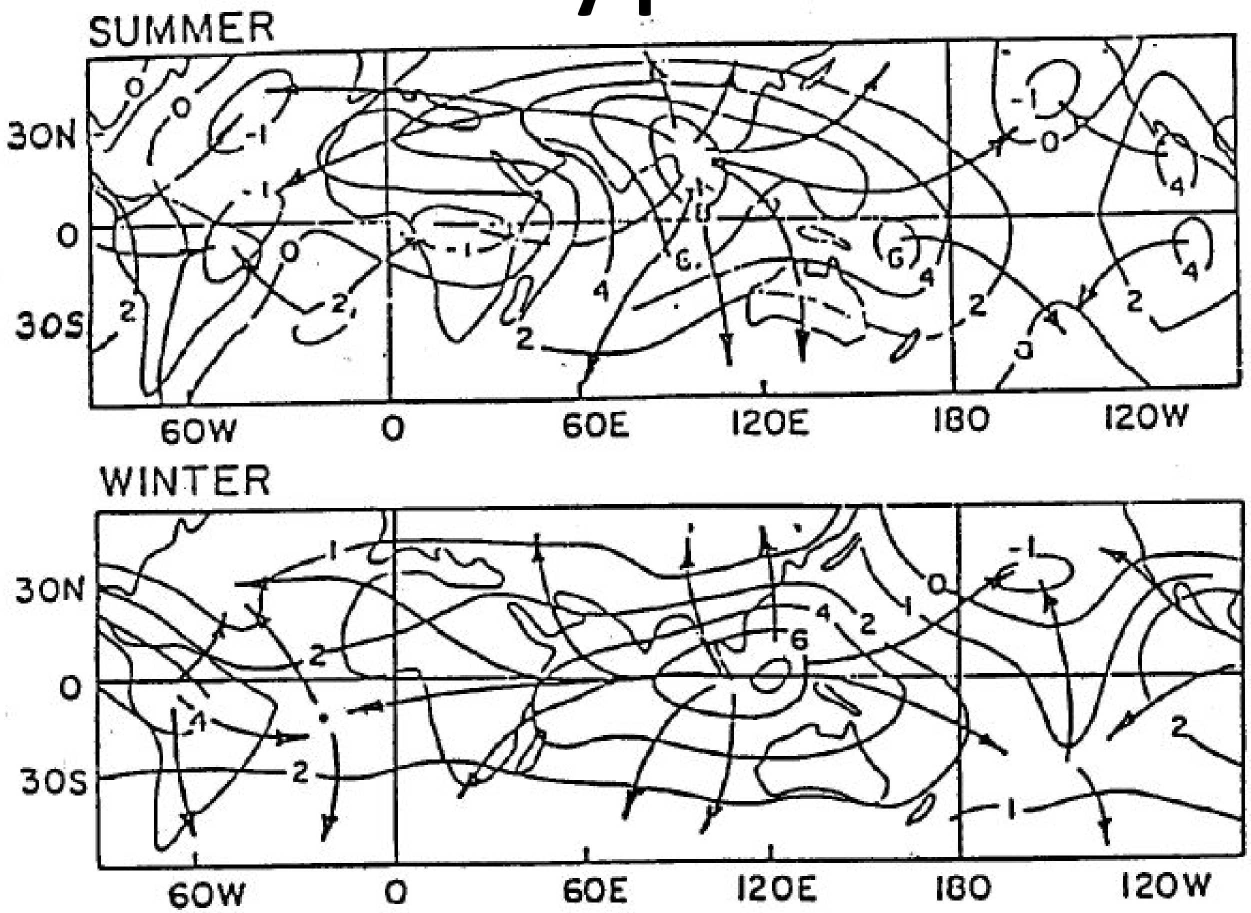
$\nabla^2\psi>0\rightarrow$ the slope of ψ increases in all directions, ψ being less than the average of the local value.

$\nabla^2\psi<0\rightarrow$ the slope of ψ decreases in all directions, ψ being locally greater than the average value.

$$\zeta=\nabla^2\psi\approx-\psi\Rightarrow\begin{cases}\nabla^2\psi>0\rightarrow\psi_{i,j}^{\min}<0\\ \nabla^2\psi<0\rightarrow\psi_{i,j}^{\max}>0\end{cases}$$



Velocity potential



When interpreting the χ-fields, a note of caution is appropriate. Remember that $\nabla^2 \chi = \nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}}$ and that $|w|$ is proportional to $\left|\nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}}\right|$. Therefore centres of χ maximum or minimum do not coincide with centres of w maximum or minimum.

$\Rightarrow$ The latter occur where $\nabla^2 \chi$ is a maximum or minimum.

Relationship between divergence and vertical motion

The continuity equation (in isobaric coordinates) is given by: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$

$$\frac{\partial \omega}{\partial p} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_H = -\nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}} = -\delta$$

$\Leftrightarrow$ the velocity divergence on an isobaric surface is related to how vertical velocity changes with height.

Let us now integrate this between two arbitrary isobaric surfaces p_B and p_T,

where $p_B > p_T$ (i.e., p_B is found closer to the surface than is p_T given that pressure decreases with increasing altitude)

$$\int_{p_B}^{p_T} \delta dp = -\int_{p_B}^{p_T} \frac{\partial \omega}{\partial p} dp = -\int_{p_B}^{p_T} d\omega = \omega(p_B) - \omega(p_T)$$

$\omega \approx -\rho g w$

$\omega < 0 \Leftrightarrow w > 0$

$\Rightarrow$ the difference in vertical motion ω over some vertical layer bounded by two isobaric levels p_B and p_T (where $p_B > p_T$) is equal to the vertically integrated divergence within that layer

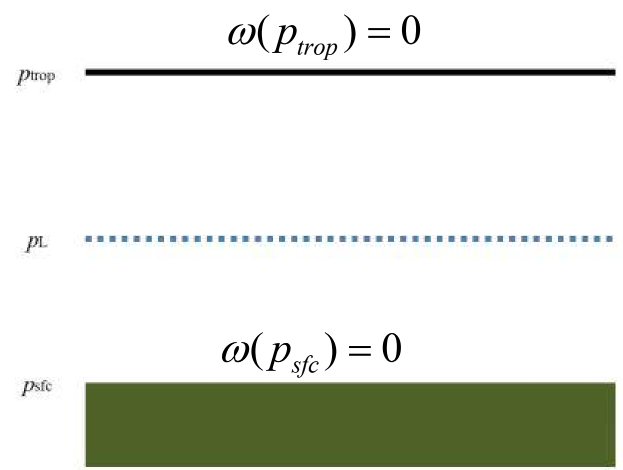
$$\omega(p_B) - \omega(p_T) = -\int_{p_B}^{p_T} \delta dp$$

Dines compensation and level of non-divergence

Let us now consider a hypothetical atmosphere comprised of two layers:

one between the surface $\left(p_{sfc}\right)$ and some mid-tropospheric isobaric level $\left(p_L\right)$,

and one between some mid-tropospheric isobaric level $\left(p_L\right)$ and the tropopause $\left(p_{trop}\right)$.



For the lower layer, $\int_{p_{sfc}}^{p_L} \delta dp = \omega(p_{sfc}) - \omega(p_L)$

The surface is a rigid bound on vertical motions, so, $\boxed{\int_{p_{sfc}}^{p_L} \delta dp = -\omega(p_L)}$

For the upper layer, $\int_{p_L}^{p_{trop}} \delta dp = \omega(p_L) - \omega(p_{trop})$

Due to the large atmospheric stability found at the tropopause, the tropopause itself is also treated as a rigid bound on vertical motion. therefore $\omega(p_{trop}) = 0$,

thus simplifies to $\boxed{\int_{p_L}^{p_{trop}} \delta dp = \omega(p_L)}$

Upon inspection, we can notice, $\boxed{-\int_{p_{sfc}}^{p_L} \delta dp = \int_{p_L}^{p_{trop}} \delta dp}$

$$\boxed{\begin{array}{l} \omega \approx -\rho g w \\ \omega < 0 \Leftrightarrow w > 0 \end{array}}$$

Dines compensation and level of non-divergence

In other words, the vertically integrated divergence in the lower layer is cancelled out by the vertically integrated divergence in the upper layer.

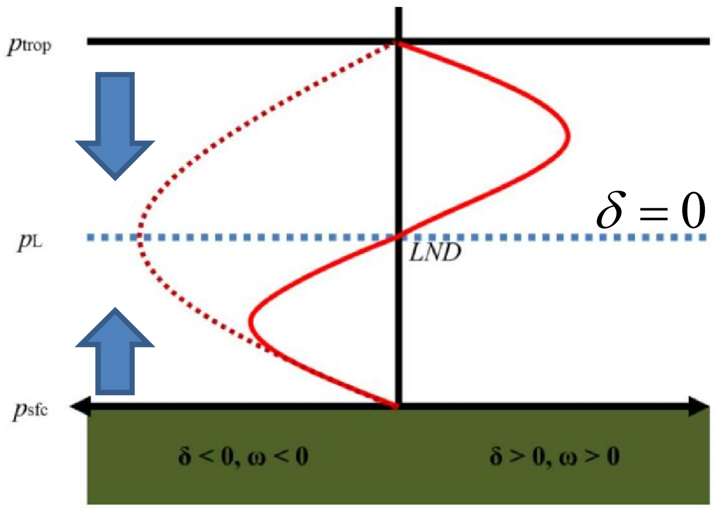
$$\int_{p_{sfc}}^{p_L} \delta dp + \int_{p_L}^{p_{trop}} \delta dp = 0$$

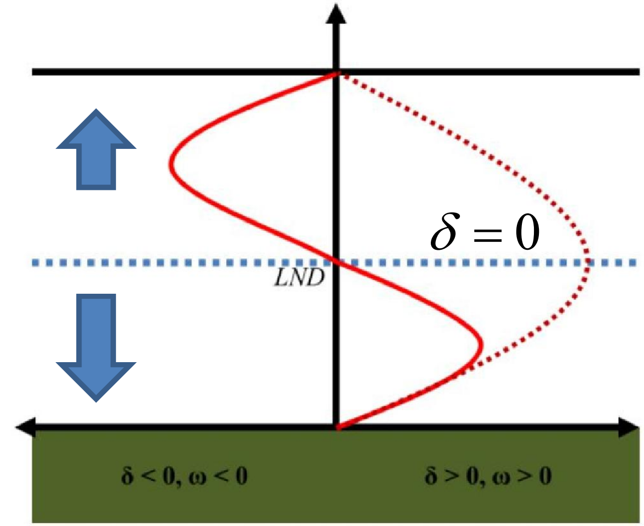
Stated differently, the divergence within the lower layer is equal in magnitude and opposite in sign to the divergence in the upper layer. This implies that the two are in balance with each other, such that one compensates for the other. This important principle is known as Dines' compensation principle

Thus, an important corollary to Dines' compensation principle states that there must be at least one level at which the divergence is zero.

$$\Leftrightarrow \text{ where } \frac{\partial \omega}{\partial p} = 0 \rightarrow \omega_{\max}$$

$\Rightarrow$ This level is the level of non-divergence. In the troposphere, we often find a level of non-divergence in the middle troposphere, typically between 500 – 600 hPa.



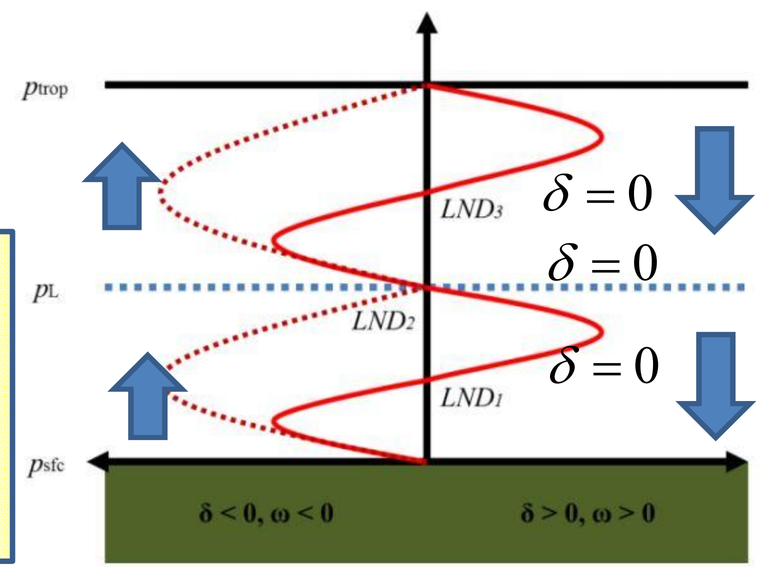


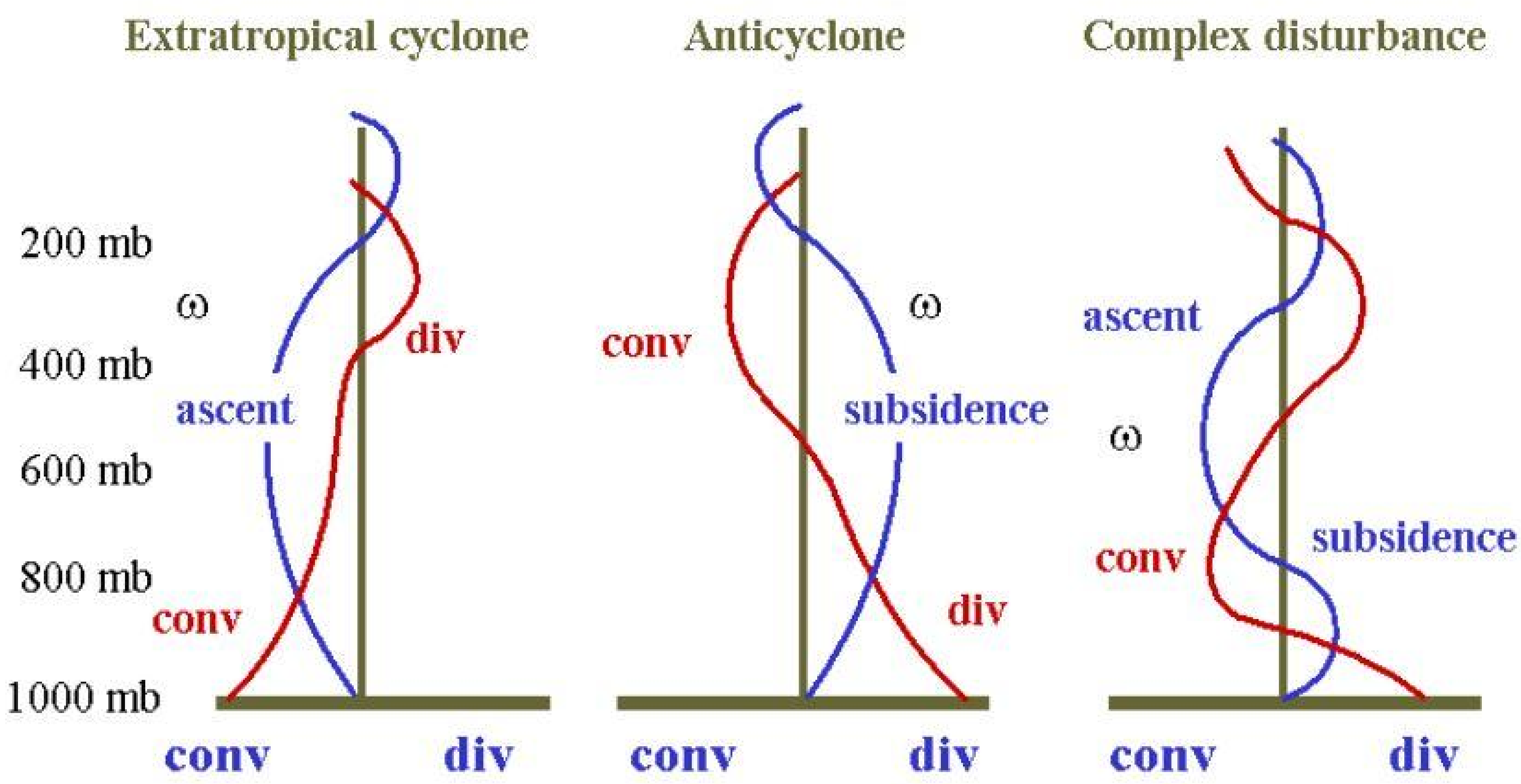
Horizontal divergence (solid red line) and vertical motion (dashed red line). LND = Level of non-divergence

The real atmosphere typically cannot be considered by two vertical layers. The vertical motion at any isobaric level p is equal to the negative of the integrated divergence between the surface and $p \Rightarrow$

$$\boxed{\omega(p) = -\int_{p_{sfc}}^{p} \delta dp}$$

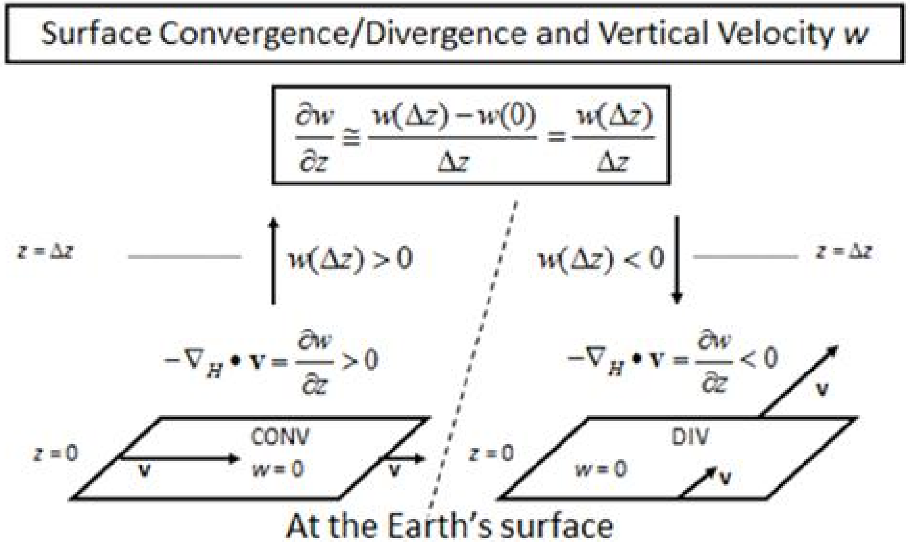
In real atmosphere, there would be multiple LND's wherever there is a shift from divergence to convergence, or ascent is maximized.

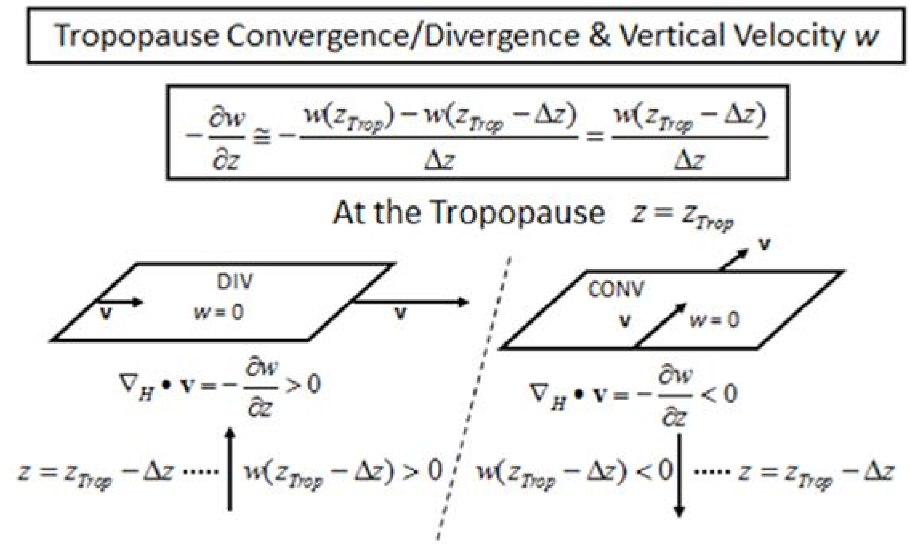


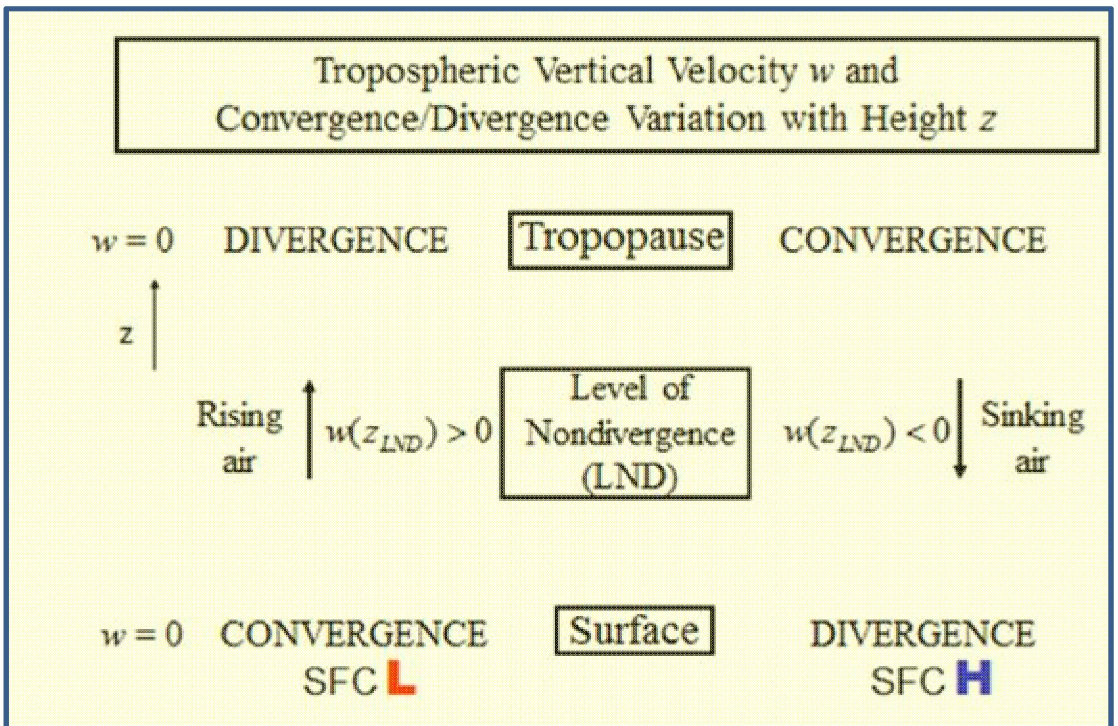


Vertical distribution of vertical motion and horizontal divergence in an extra- tropical cyclone, an anticyclone and in a more complex synoptic-scale disturbance, after (Sutcliffe, 1947).

Convergence/divergence and vertical motion





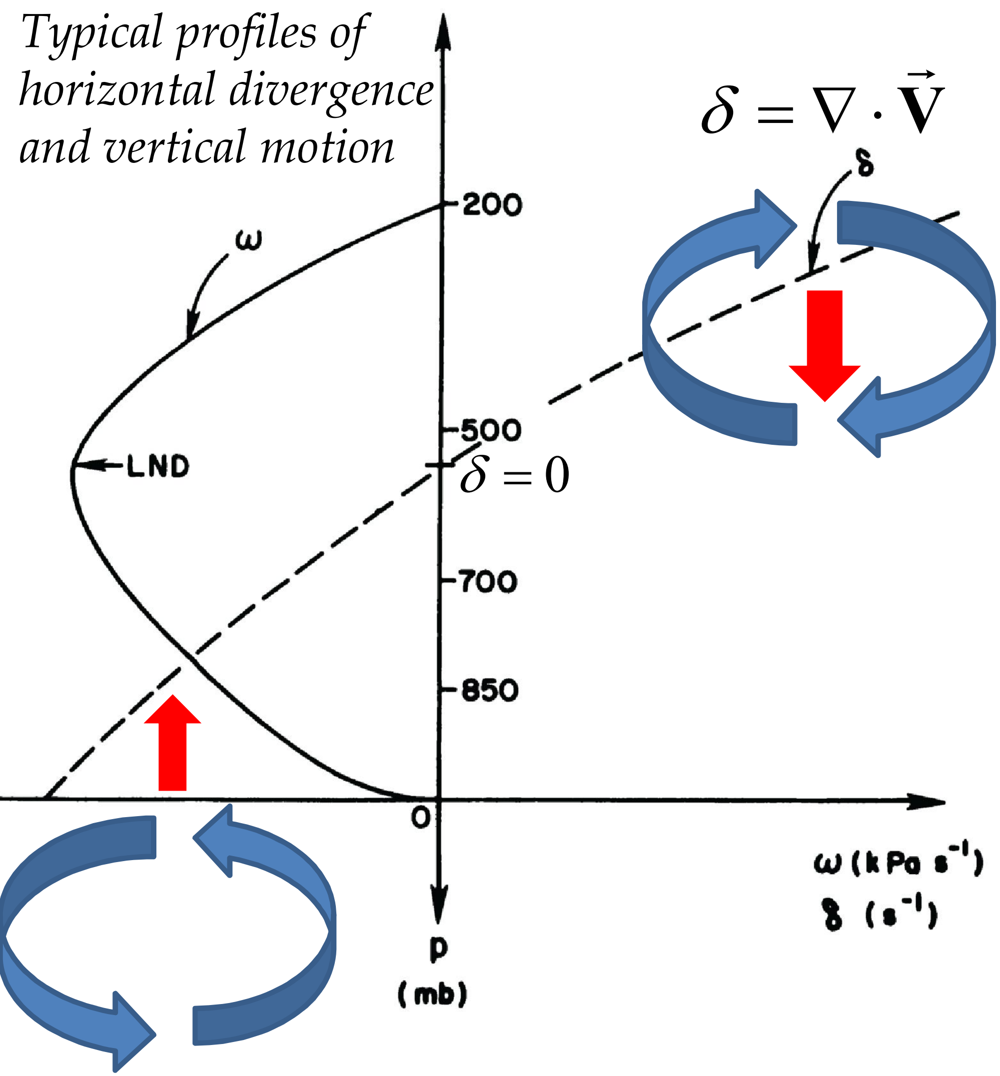


$$\frac{\partial w}{\partial z} = -\delta = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_z$$

$$\frac{\partial w}{\partial z} = 0 \text{ where } w = w_{max} \text{ and } \delta = 0$$

This level is level of non-divergence. It is found on average near 550-600 mb.

Vertical motion



$$\frac{\partial \omega}{\partial p} = -\delta = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p \qquad (1)$$

ω is constrained to be zero at the ground and at the tropopause.

If ω is nonzero, its sign is often the same at all levels in a column in the troposphere.

Then the sign of $\frac{\partial \omega}{\partial p}$ must reverse at some level. At this level,

$$\frac{\partial \omega}{\partial p} = 0 \quad \Rightarrow \quad \text{From (1)} \quad \delta = 0$$

This level is level of non-divergence.

It is found on average near 550-600 mb.

!!! A quantity that is not routinely measured !!!!
Always inferred

Pressure tendency and Dines decomposition

- ***Tendency: How a variable changes with time at a particular point or locally, e.g., pressure tendency.***
- ***Pressure in a column is due to the weight of the air above***
- ***Lower the mass of the air above, lower the pressure***

The hydrostatic equation is $dp = -\rho g dz$

$$p = g\int_{z_1}^{z_2} \rho dz \ \ \{z_2 = 0 \text{ top of the atmosphere}\}$$

Local pressure tendency for a given air column with thickness dz (assume the thickness is constant)

$$\left(\frac{\partial p}{\partial t}\right)_z = g dz \int_z^{\infty} \frac{\partial \rho}{\partial t}$$

$$\text{Pressure tendency: } \left(\frac{\partial p}{\partial t}\right)_z = g\int_z^\infty \frac{\partial \rho}{\partial t} dz$$

Let us consider the mass continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \vec{\mathbf{V}}\right) = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$$

As $\left(\frac{\partial p}{\partial t}\right)_z = g\int_z^\infty \left(\frac{\partial \rho}{\partial t}\right)_z dz \quad \left[\frac{\partial p}{\partial z} = -\rho g \rightarrow \frac{\partial p}{\partial t} = -\rho g \frac{\partial z}{\partial t}\right]$

$$\left(\frac{\partial p}{\partial t}\right)_z = -g\left\{\int_z^\infty \frac{\partial}{\partial x}(\rho u) dz + \int_z^\infty \frac{\partial}{\partial y}(\rho v) dz\right\} - g\left\{\int_z^\infty \frac{\partial}{\partial z}(\rho w) dz\right\}$$

$$\left(\frac{\partial p}{\partial t}\right)_z = -g\left\{\int_z^\infty \frac{\partial}{\partial x}(\rho u) dz + \int_z^\infty \frac{\partial}{\partial y}(\rho v) dz\right\} - g\int_z^\infty d(\rho w)$$

$$\left(\frac{\partial p}{\partial t}\right)_z = -g\left\{\int_z^\infty \frac{\partial}{\partial x}(\rho u) dz + \int_z^\infty \frac{\partial}{\partial y}(\rho v) dz\right\} - \cancel{g[\rho w]_{z=0}^{z=\infty}}$$

$$\left\{\begin{array}{l} w = 0 \text{ near the ground} \\ w = 0 \text{ at } z = \infty \end{array}\right\}$$

$$\left(\frac{\partial p}{\partial t}\right)_z = -g\left\{\int_z^\infty \frac{\partial}{\partial x}(\rho u) dz + \int_z^\infty \frac{\partial}{\partial y}(\rho v) dz\right\}$$

Local tendency — Horizontal mass convergence

$$\text{Pressure tendency: } \left(\frac{\partial p}{\partial t}\right)_z = g\int_z^\infty \frac{\partial \rho}{\partial t} dz$$

$$\left(\frac{\partial p}{\partial t}\right)_z = -g\left\{\int_z^\infty \frac{\partial}{\partial x}(\rho u)\,dz + \int_z^\infty \frac{\partial}{\partial y}(\rho v)\,dz\right\}$$

Local tendency — Horizontal mass convergence

Remembering that 90% of the mass of the atmosphere lies beneath the tropopause and that we are treating the density as a mean density for the air column (a constant), then it can be seen that, at a synoptic-scale, surface pressure tendencies are directly related to horizontal divergence patterns in the troposphere.

$\Rightarrow$ For synoptic scale, one can write $$\frac{\partial p}{\partial t} = -\rho g \int_1^2 \nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}}\, dz$$

Dines compensation principle

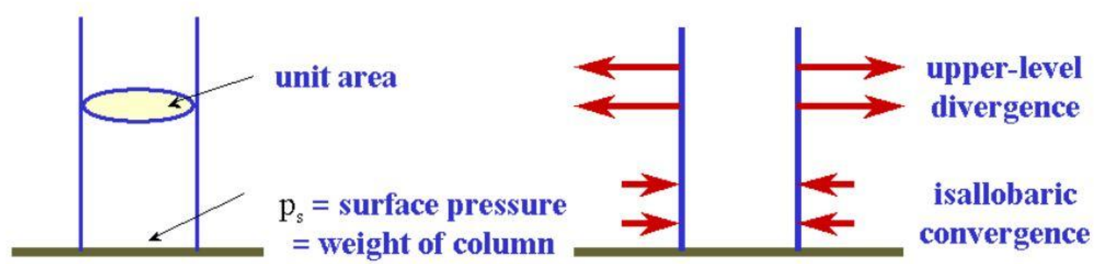
Large scale motions in the atmosphere are in close hydrostatic balance. Hence the pressure at the base of a fixed column of air is proportional to the mass of air in that column; if the total mass decreases, so will the surface pressure, and vice versa

Dines showed that low-level convergence is very nearly equal to the divergence at upper levels and pointed out that upper divergence must exceed the low-level convergence when a low deepens

Surface pressure tendency equation: $\frac{\partial p_s}{\partial t} = -g\int_0^\infty \nabla \cdot \left(\rho \vec{\mathbf{V}}\right) dz$

Geostrophic: $\rho \vec{\mathbf{V}} = \frac{1}{f_0}\hat{\mathbf{k}} \times \nabla p$

$\nabla \cdot \left(\rho \vec{\mathbf{V}}\right) = 0 \Leftrightarrow \frac{\partial p_s}{\partial t} = 0$



⇔ It follows that any local change in surface pressure is associated entirely with ageostrophic motion.

Computation of ω-adiabatic method

Alternative: Quasi-geostrophic ω

A second method for inferring vertical velocities, which is not so sensitive to errors in the measured horizontal velocities, is based on the thermodynamic energy equation:

$$\boxed{\frac{dT}{dt} - S_p\omega = \frac{\dot{Q}}{c_p}}, \qquad \text{where } S_p \equiv \text{static stability} = \left(\frac{\gamma_{dry} - \gamma_{ambient}}{\rho g}\right), \quad \gamma = \text{lapse rate}$$

If the diabatic heating $\dot{Q}$ is small compared to other terms in the heat balance, we can write,

$$\omega \equiv \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{1}{S_p}\left(\frac{\partial T}{\partial t} + \vec{\mathbf{V}}_{\mathbf{H}} \cdot \vec{\nabla} T\right)$$

⇒ Because temperature advection can usually be estimated quite accurately in midlatitudes by using geostrophic winds, the adiabatic method can be applied when only geopotential and temperature data are available.

A disadvantage of the adiabatic method is that the local rate of change of temperature is required.

Is this method suitable for the tropical monsoon region? – weak temperature gradients and strong diabatic heating are characteristics of monsoon region.

Unless observations are taken at close intervals in time, it may be difficult to accurately estimate $\frac{\partial T}{\partial t}$ over a wide area.

This method is also rather inaccurate in situations where strong diabatic heating is present, such as storms in which heavy rainfall occurs over a large area.

Vorticity equation (Cartesian form) – z coordinates

Absolute vorticity $= \zeta + f = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f$

Recall: $\frac{\partial}{\partial t}\left\{\hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{V}}\right)\right\} = \frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \alpha\frac{\partial p}{\partial x} + F_x\right) = 0 \quad (1)$$

specific volume $\alpha = \frac{1}{\rho}$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \alpha\frac{\partial p}{\partial y} + F_y\right) = 0 \quad (2)$$

$\frac{\partial f}{\partial y} = \frac{1}{a}\frac{\partial}{\partial \varphi}\left(2\Omega \sin\varphi\right) = \frac{2\Omega\cos\varphi}{a} \equiv \beta$

$a =$ radius of the Earth

Subtracting (1) from (2),

$$\frac{\partial \zeta}{\partial t} + \vec{\mathbf{V}} \cdot \vec{\nabla}\zeta + w\frac{\partial \zeta}{\partial z} + \boxed{\beta} v = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) - \left(\frac{\partial \alpha}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y}\frac{\partial p}{\partial x}\right) + \hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{F}}_{fric}\right)$$

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \hat{\mathbf{k}} \cdot \left(\nabla p \times \nabla \alpha\right) + \hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{F}}_{fric}\right)$$

Stretching — Tilting — Baroclinicity — Friction

$\Rightarrow$ Rate of change of absolute vorticity following the motion

Non-divergent vorticity equation (RHS $\equiv 0$)
(valid for barotropic atmosphere)

$$\Rightarrow \frac{\partial \zeta}{\partial t} + \vec{\mathbf{V}}_{\mathbf{H}} \cdot \nabla\left(\zeta + f\right) = 0$$

Non-divergent barotropic vorticity equation

In the case of a barotropic fluid (purely horizontal flow, $w = 0$)
with constant fluid depth H

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \hat{\mathbf{k}}\cdot(\nabla p \times \nabla\alpha) + \hat{\mathbf{k}}\cdot\left(\nabla\times\vec{\mathbf{F}}_{fric}\right)$$

$$\frac{d}{dt}(\zeta + f) = 0 \quad \Rightarrow$$ absolute vorticity conservation following the horizontal motion.

For horizontal motion, that is non-divergent, the flow field can be represented by stream function (ψ) such that

$$u_\psi = -\frac{\partial\psi}{\partial y}, \quad v_\psi = \frac{\partial\psi}{\partial x}$$

In the absence of sources such as stretching, tilting, baroclinicity, friction (non-divergent, barotropic, inviscid fluid)

Absolute vorticity is conserved following the motion

$$\frac{d_H(\zeta + f)}{dt} = \frac{\partial\zeta}{\partial t} + u_\psi\frac{\partial\zeta}{\partial x} + v_\psi\frac{\partial\zeta}{\partial x} + v_\psi\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) - \left(\frac{\partial\psi}{\partial y}\frac{\partial\varsigma}{\partial x}\right) + \left(\frac{\partial\psi}{\partial x}\frac{\partial\varsigma}{\partial y}\right) + \frac{\partial\psi}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{a}\frac{\partial}{\partial\varphi}(2\Omega\sin\varphi) = \frac{2\Omega\cos\varphi}{a} \equiv \beta$$

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \left(\frac{\partial\psi}{\partial x}\frac{\partial\varsigma}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\varsigma}{\partial x}\right) + \beta\frac{\partial\psi}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\nabla^2\psi\right) + J(\psi,\varsigma) + \beta\frac{\partial\psi}{\partial x} = 0$$

The flow in the mid-troposphere is often nearly nondivergent on the synoptic scale, the barotropic vorticity equation provides a surprisingly good model for short-term forecasts of the synoptic-scale 500-hPa flow field.

Linear model

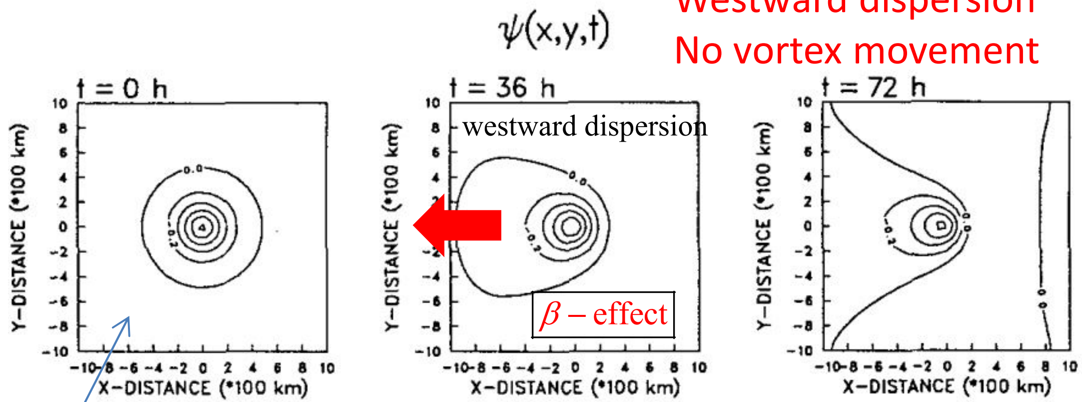
$$\zeta = \nabla^2\psi$$

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \beta\frac{\partial\psi}{\partial x} = 0$$

Linear

Westward dispersion

No vortex movement



b = 1.0 r_m = 100 km v_m = 40 m s^{-1}

1 May 1987 JOHNNY C. L. CHAN AND R. T. WILLIAMS 1257

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Analytical and Numerical Studies of the Beta-Effect in Tropical Cyclone Motion. Part I: Zero Mean Flow

Johnny C. L. Chan* and R. T. Williams

Department of Meteorology, Naval Postgraduate School, Monterey, CA 93943

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$$V(r) = V_m\left(\frac{r}{r_m}\right)\exp\left\{\frac{1}{b}\left[1-\left(\frac{r}{r_m}\right)^b\right]\right\}$$

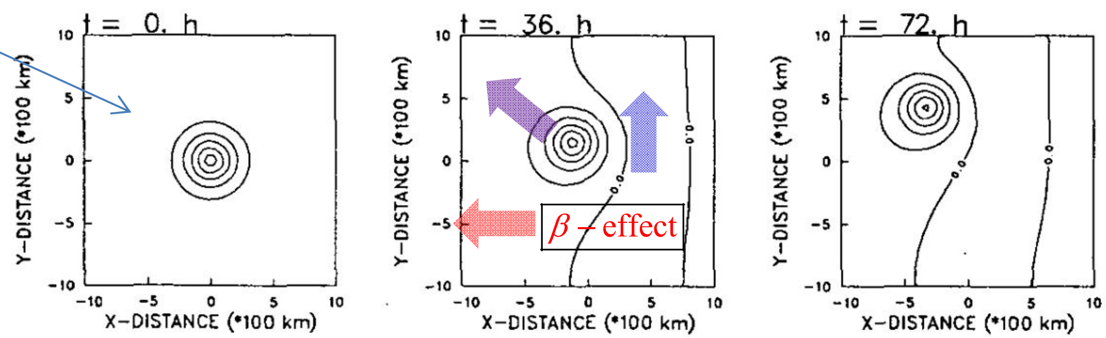
$$\zeta(r) = \frac{2V_m}{r_m}\left[1-\frac{1}{2}\left(\frac{r}{r_m}\right)^b\right]\exp\left\{\frac{1}{b}\left[1-\left(\frac{r}{r_m}\right)^b\right]\right\}$$

Initial conditions

Axi-symmetric vortex

(no radial wind)

STREAMFUNCTION (NONLINEAR MODEL)



b = 1.0 r_m = 100.0 km v_m = 40.0 m s^{-1}

Northwest movement of the vortex

Non-linear model

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + J(\psi,\varsigma) + \beta\frac{\partial\psi}{\partial x} = 0$$

Nonlinear Linear

Divergence (stretching/shrinking) term

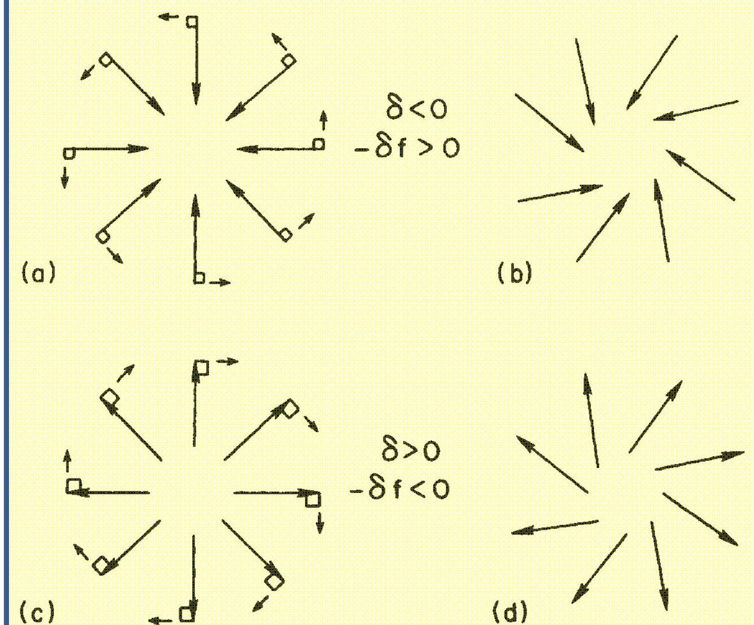
The divergence term represents the fluid analog to conservation of angular momentum in rigid-body mechanics.

This represents the stretching $\left(\frac{dw}{dz} > 0\right)$ or shrinking $\left(\frac{dw}{dz} < 0\right)$ of an air column and its effects on Earth's vorticity and relative vorticity.

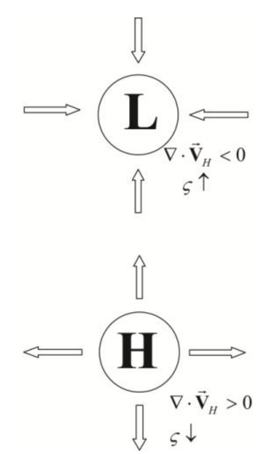
Its effects are analogous to the increase and decrease in rotational speed experienced by an ice skater whose arms are brought in or are extended outward;

A purely convergent flow field acquires cyclonic relative vorticity from the Earth's vorticity as the air is accelerated to the right in the Northern Hemisphere by the Coriolis force (and to the left in the Southern Hemisphere). Similarly a purely divergent flow acquires anticyclonic vorticity.

Coriolis force (small vectors), convergent/ divergent wind field (large vectors)



$$-\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{dw}{dz}$$

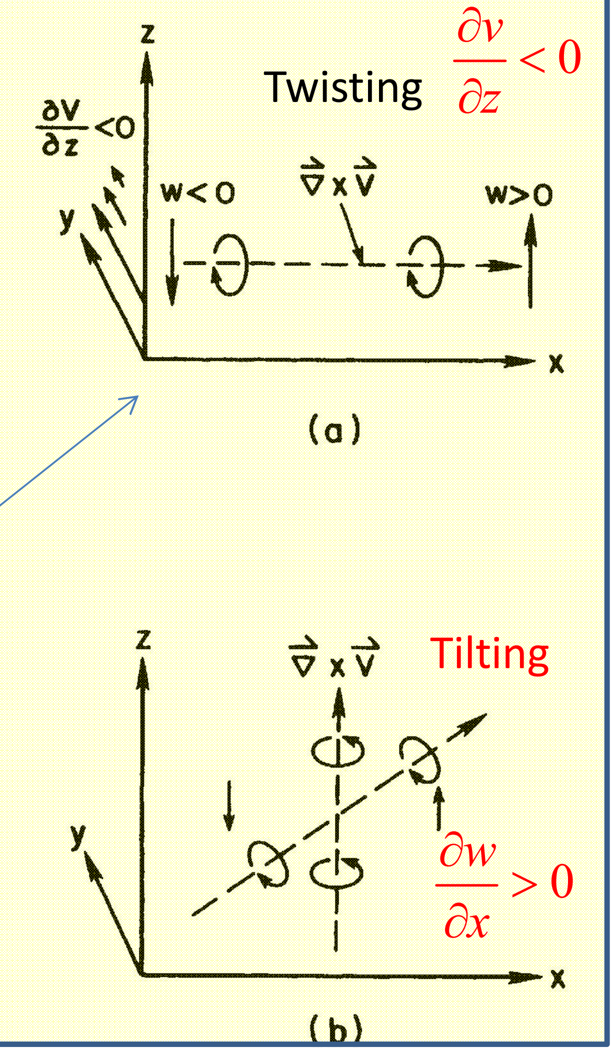


$$\frac{d\zeta}{dt} = -(\varsigma + f)\,\delta - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \hat{\mathbf{k}}\cdot(\nabla p \times \nabla\alpha) + \hat{\mathbf{k}}\cdot\left(\nabla\times\vec{\mathbf{F}}_{fric}\right)$$

Tilting/twisting

(a) Suppose that initially the vorticity vector $(\nabla \times \vec{\mathbf{V}})$ points in the x - direction owing to a decrease in v with height. Suppose also that there is a rising motion and sinking motion at large and small values of x, respectively. Then the vorticity vector will become tilted about the y-axis,

(b) so that the vorticity vector becomes more aligned with the z - axis. Thus, the vorticity about the x-axis has been converted into vertical vorticity.



Mathematically, the tilting term in this case,

$$\hat{\mathbf{k}} \cdot \left(\frac{\partial \vec{\mathbf{V}}}{\partial z} \times \nabla w \right) = -\left(\frac{\partial v}{\partial z} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \frac{\partial w}{\partial y} \right)$$

$$\frac{d\zeta}{dt} = -(\zeta + f)\nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}} - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} \right) + \hat{\mathbf{k}} \cdot (\nabla p \times \nabla \alpha) + \hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{F}}_{fric} \right)$$

Vertical shear in the horizontal motion is going to twist

Horizontal shear in the vertical motion is going to tilt

Critical for tornadoes

Vorticity equation – Tilting/twisting term

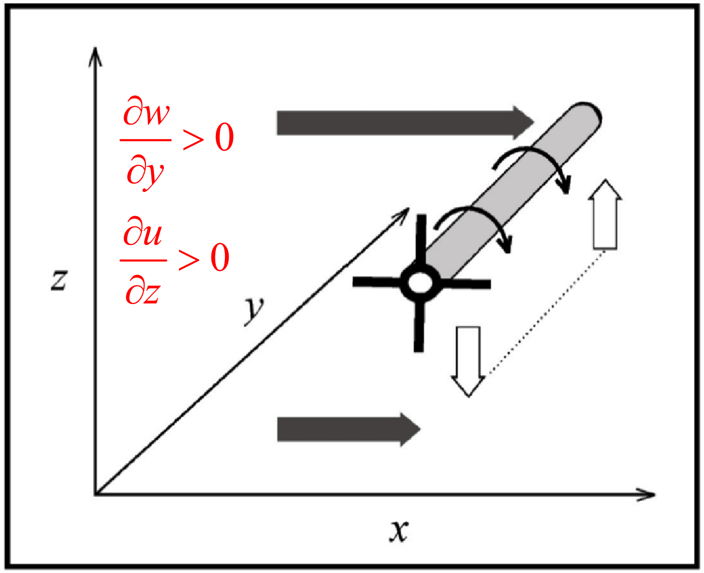
Vertical shear in the horizontal motion is going to twist
Horizontal shear in the vertical motion is going to tilt

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

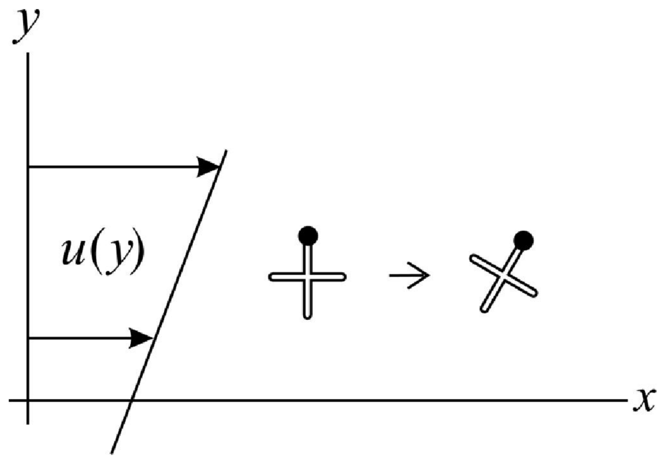
$$\frac{\partial w}{\partial y}>0,\quad \frac{\partial u}{\partial z}>0$$

$$\frac{d}{dt}(\varsigma+f)>0$$

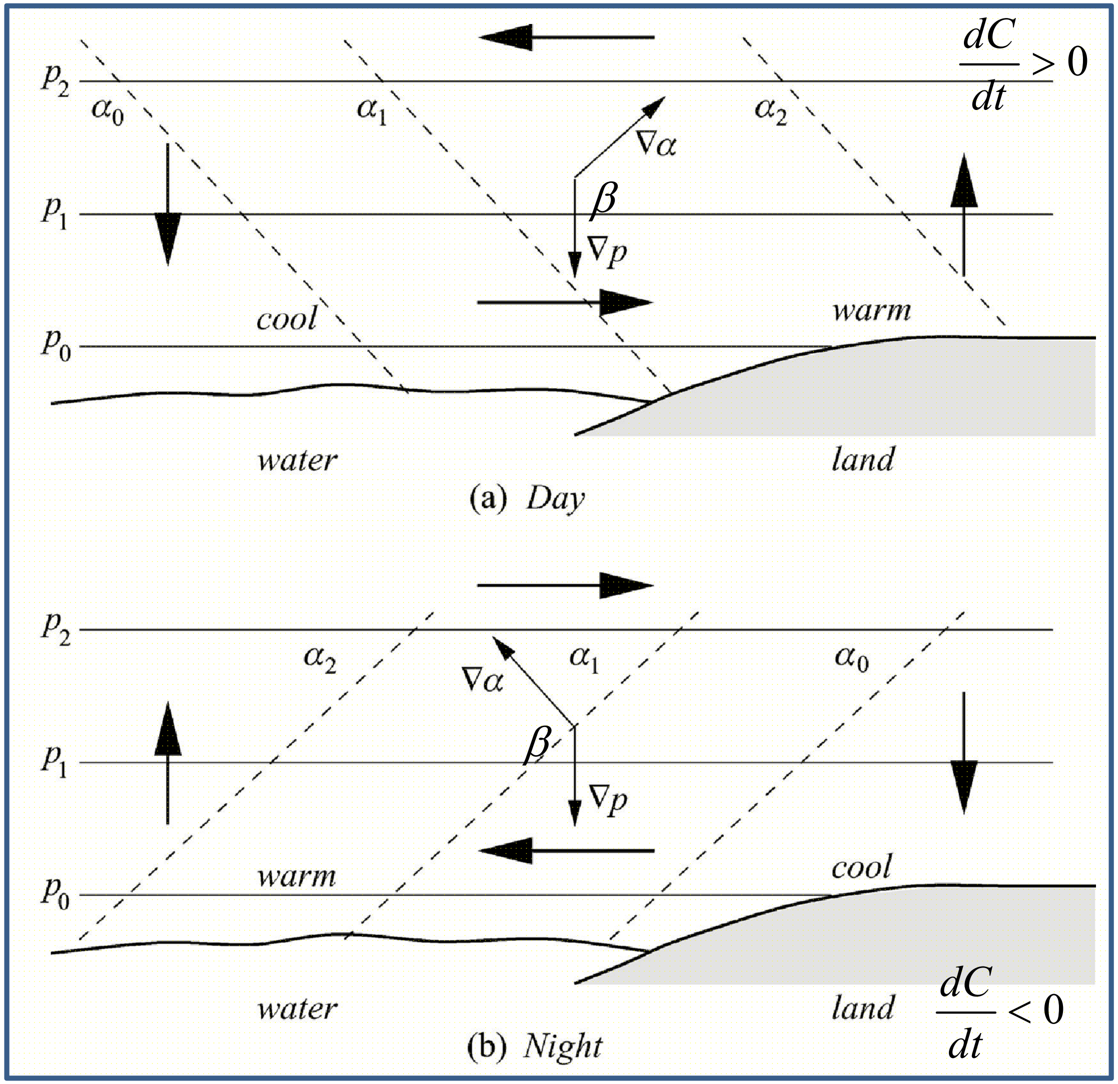
twisting **arises** due to vertical shear in the horizontal motion, and horizontal shear in the vertical motion



$$\frac{d}{dt}(\zeta+f)=-(\zeta+f)\nabla\cdot\vec{\mathbf{V}}-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\hat{\mathbf{k}}\cdot(\nabla p\times\nabla\alpha)+\hat{\mathbf{k}}\cdot\left(\nabla\times\vec{\mathbf{F}}_{fric}\right)$$



Solenoidal or baroclinicity



$$\frac{dC}{dt} = -\iint_A (\nabla\alpha \times \nabla p) \cdot d\vec{\mathbf{A}}$$

Solenoidal term: $\frac{dC}{dt} = -|\nabla\alpha||\nabla p|\sin\beta$

If the atmosphere is barotropic

$\hat{\mathbf{k}} \cdot (\nabla p \times \nabla\alpha) = 0$

Pressure and density surfaces lie on each other

If the atmosphere is baroclinic

$\hat{\mathbf{k}} \cdot (\nabla p \times \nabla\alpha) > 0 \Rightarrow$ cyclonic

$\hat{\mathbf{k}} \cdot (\nabla p \times \nabla\alpha) < 0 \Rightarrow$ anticyclonic

Pressure and density surfaces do not lie on each other. Solenoidal term acts as a circulation mechanism to bring the pressure and density surfaces lie on each other

In general the circulation that develops would be such that the density and pressure surfaces would become parallel (baroclinic ⟹ barotropic)

$$\frac{d}{dt}(\varsigma + f) = -(\varsigma + f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}} - \hat{\mathbf{k}}\cdot\left(\frac{\partial\vec{\mathbf{V}}}{\partial z}\times\nabla w\right) + \hat{\mathbf{k}}\cdot(\nabla p\times\nabla\alpha) + \hat{\mathbf{k}}\cdot\left(\nabla\times\vec{\mathbf{F}}_{friction}\right)$$

Stretching — Tilting — Solenoidal — Friction

Effect of friction

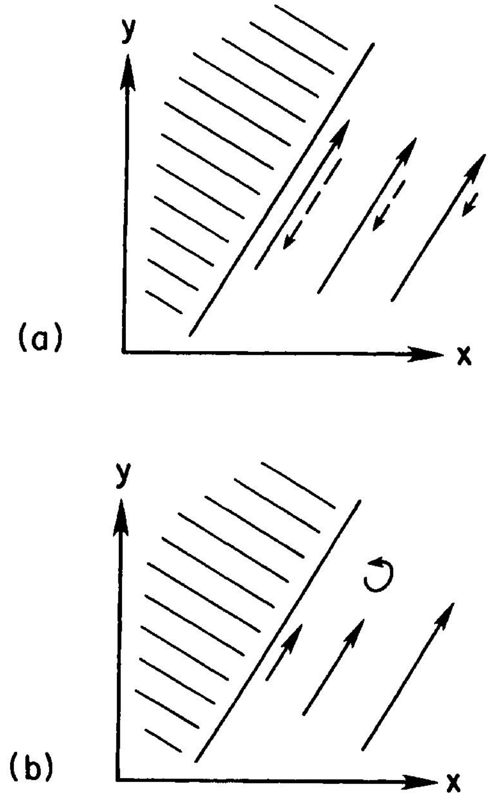
Illustration of frictional generation of vorticity alongside a wall

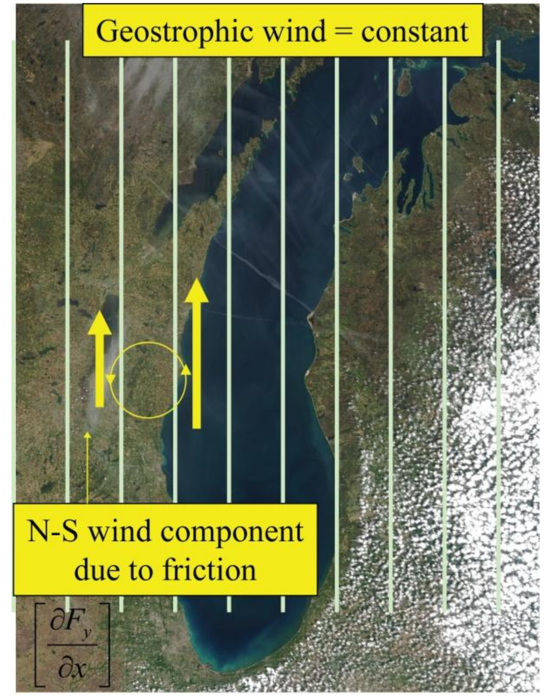
(a) Air is forced to flow (solid-line vectors) along the edge of a vertical wall (hatched area). Friction (dashed vectors) acts in the direction opposite to that of the wind; the magnitude of the friction force is greatest along the edge of the wall, and decreases in magnitude with distance from the wall. Thus the wind is slowed down the most along the wall.

(b) and least away from the wall.

In this figure, $\hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{F}}_{fric} \right) > 0$

Example: western boundary currents in the ocean





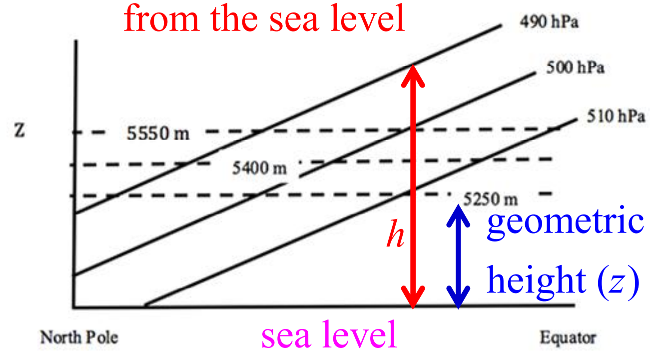
$$\frac{d\zeta}{dt} = -(\zeta + f)\,\nabla \cdot \vec{\mathbf{V}}_{\mathbf{H}} + \hat{\mathbf{k}} \cdot \left(\frac{\partial \vec{\mathbf{V}}}{\partial z} \times \nabla w \right) + \hat{\mathbf{k}} \cdot (\nabla p \times \nabla \alpha) + \hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{F}}_{friction} \right)$$

Stretching   Tilting   Solenoidal   Friction

Vorticity equation (Cartesian form): (x,y,p,t)

$$\text{Absolute vorticity}\Big|_p = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_p + f; \quad \omega \equiv \frac{dp}{dt}$$

geopotential height ($h \equiv \Phi/g$)
→ height of isobaric surface
from the sea level



$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} - fv + \frac{\partial \Phi}{\partial x} + F_x\right) = 0 \quad (1)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} + fu + \frac{\partial \Phi}{\partial y} + F_y\right) = 0 \quad (2)$$

Subtracting (1) from (2),

$$\frac{\partial \zeta}{\partial t} + \vec{\mathbf{V}}_H \cdot \vec{\nabla}_p \zeta + \omega\frac{\partial \zeta}{\partial p} + \beta v = -(\zeta + f)\nabla \cdot \vec{\mathbf{V}}_H + \left(\frac{\partial v}{\partial p}\frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right) + \hat{\mathbf{k}} \cdot \left(\nabla \times \vec{\mathbf{F}}_{fric}\right)$$

Absolute vorticity (p-surface) following the motion | Stretching | Tilting/Twisting | Friction

Advantage: Solenoidal/baroclinity terms are implicit

natural coordinate viewpoint

$\hat{\mathbf{s}}$ direction is parallel to flow, positive in direction of flow

$\hat{\mathbf{n}}$ direction is perpendicular to the flow, positive to left of flow.

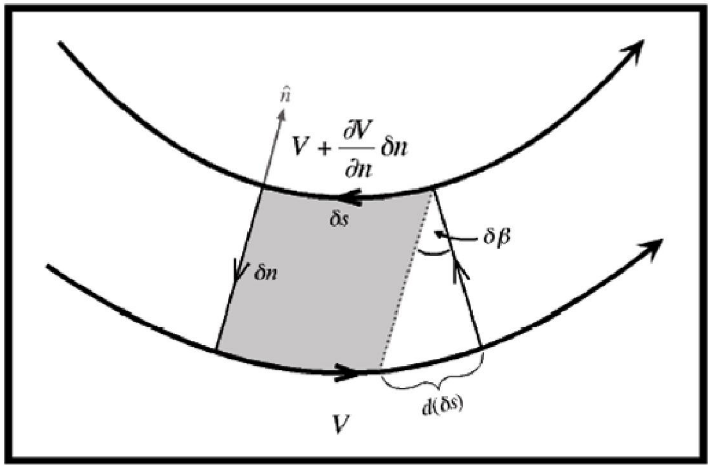
Calculate circulation

Denote the distance along the top leg as δs

Denote the distance along the bottom leg as $\delta s + d(\delta s)$

Denote the velocity along the bottom leg as V

Note: only curved sides of this box will contribute to the circulation, since the wind velocity is zero on the sides in the $\hat{\mathbf{n}}$ direction.



Using Taylor's expansion, velocity along the top: $-\left(V + \frac{\partial V}{\partial n}\delta n\right)$

$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n}$$

$$d(\delta s) = \delta\beta\delta n$$

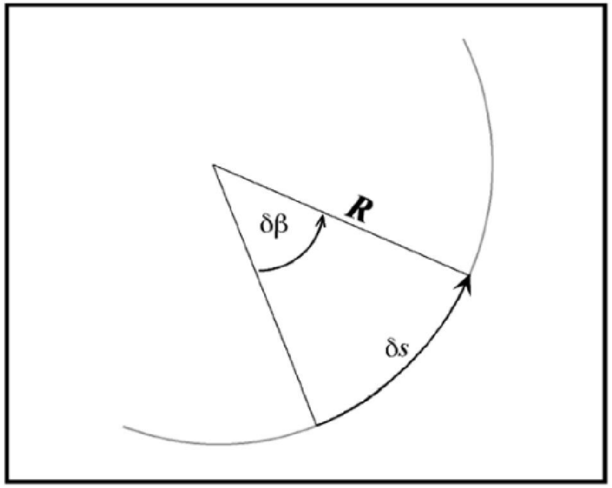
$$C = \oint \vec{\mathbf{V}} \cdot d\mathbf{l} = V(\delta s + \delta\beta\delta n) - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta s$$

$$C = V\delta\beta\delta n - \frac{\partial V}{\partial n}\delta n\delta s$$

$$\zeta = \lim_{\delta n\delta s \to 0}\left(\frac{C}{\delta n\delta s}\right) = V\frac{\partial\beta}{\partial s} - \frac{\partial V}{\partial n} = \frac{V}{R} - \frac{\partial V}{\partial n}$$

$\frac{V}{R}$ = curvature vorticity

$\frac{\partial V}{\partial n}$ = shear vorticity

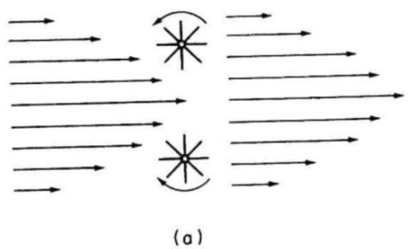


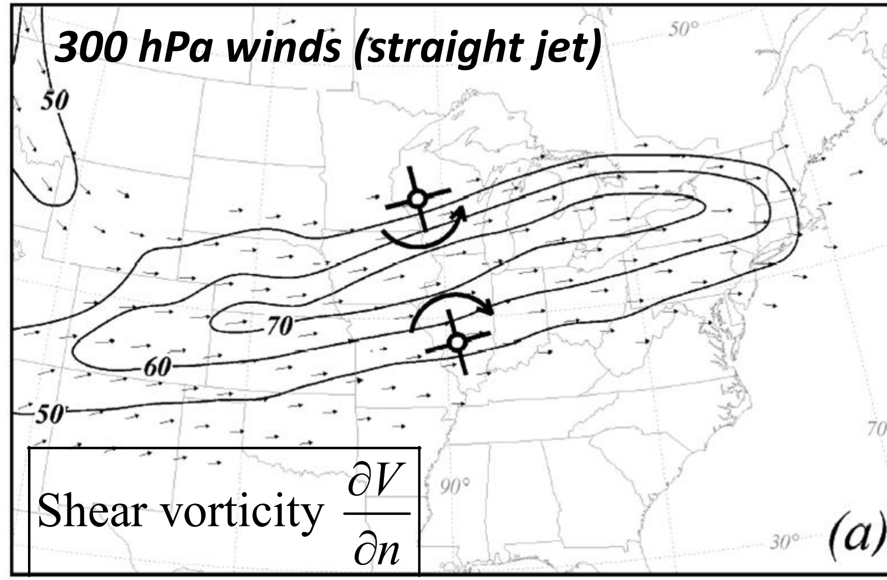
Physical interpretation

$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n}$$

Can purely straigthline flow have non-zero vorticity?

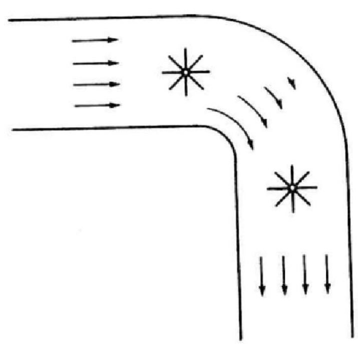
Where would this occur in the real world?

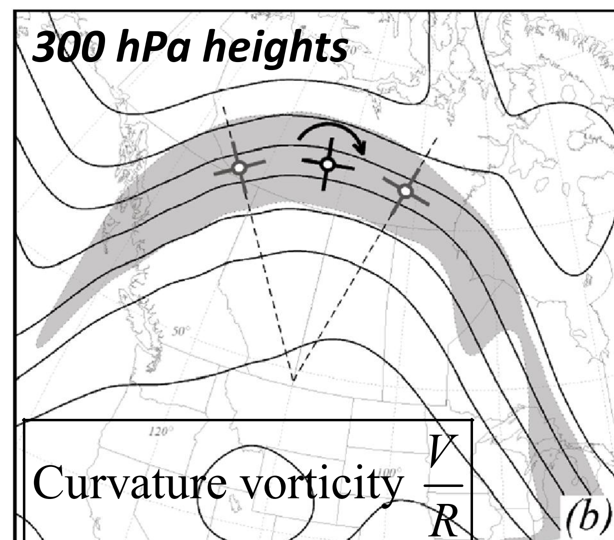


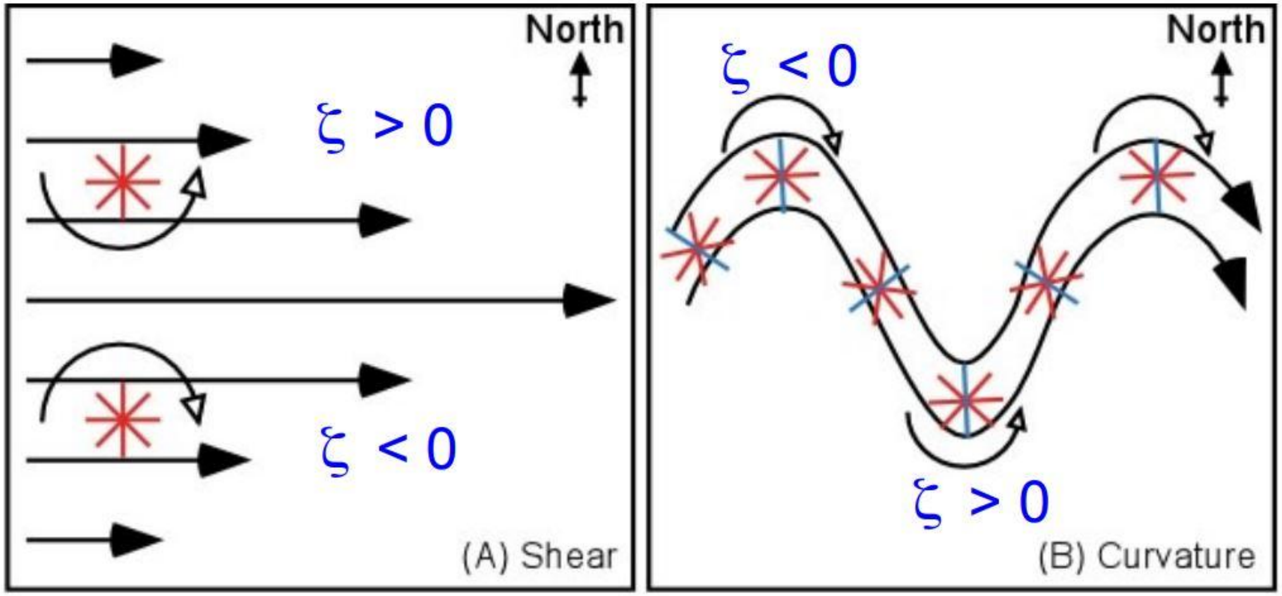


$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

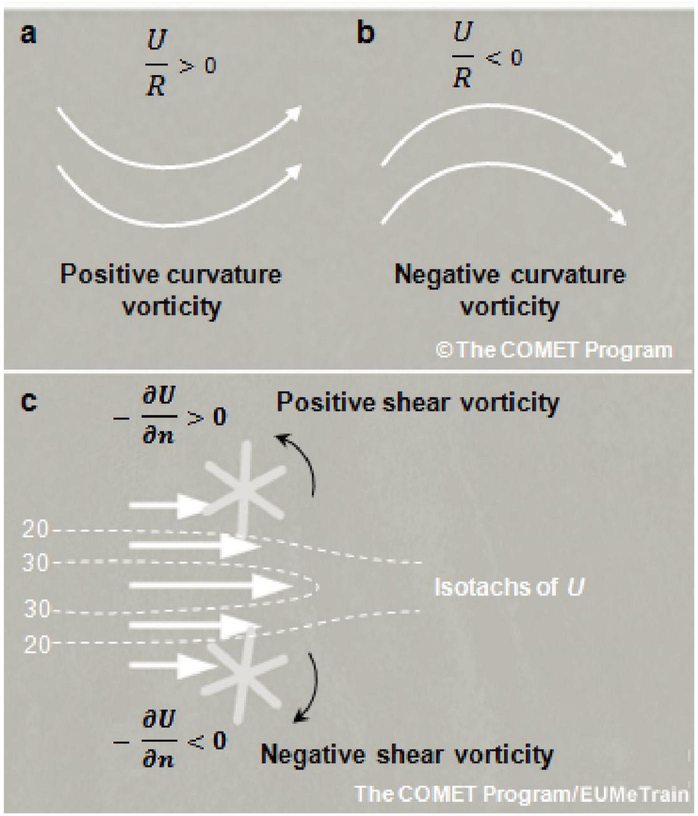
Can curved flow have zero vorticity?

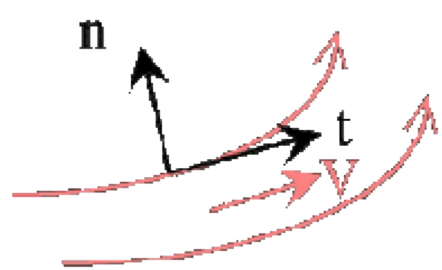




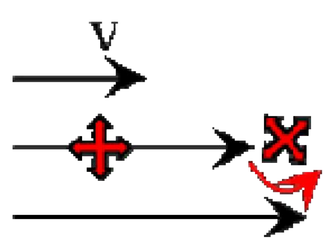


$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n}$$

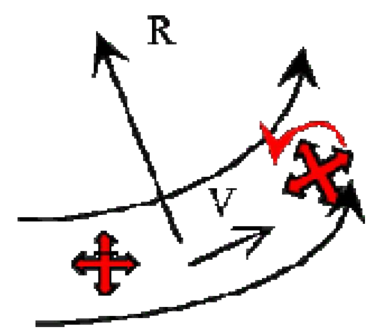




natural coordinate system



shear vorticity



curvature vorticity

Remember: For adiabatic motions

$\zeta_a = \zeta + f$ is conserved for 2D inviscid barotropic fluid motions

In stratified (baroclinic) 3D fluid, equivalent is potential vorticity

$$q \approx \frac{\zeta_a}{\rho}\left(\frac{\partial \theta}{\partial z}\right) \equiv$$

potential for generating vorticity

Potential vorticity in barotropic fluids

A model that has proved useful for elucidating some aspects of the horizontal structure of large-scale atmospheric motions is the barotropic model. In the most general version of this model, the atmosphere is represented as a homogeneous incompressible fluid of variable depth, $h(x, y, t) = z_2 - z_1$, where z_2 and z_1 are the heights of the upper and lower boundaries, respectively.

In a barotropic (incompressible) fluid, the vorticity equation (combined with the continuity equation) for can be written as:

$$\frac{d}{dt}(\varsigma+f)=-(\varsigma+f)\nabla\cdot\vec{\mathbf{V}}_{\mathbf{H}}=(\varsigma+f)\frac{\partial w}{\partial z}=(\varsigma+f)\frac{w(z_2)-w(z_1)}{h}$$

$$h\frac{d}{dt}(\varsigma+f)=(\varsigma+f)\left[\frac{dz_2}{dt}-\frac{dz_1}{dt}\right]=(\varsigma+f)\frac{dh}{dt}$$

$$\frac{1}{(\varsigma+f)}\frac{d}{dt}(\varsigma+f)=\frac{1}{h}\frac{dh}{dt} \quad\Rightarrow\quad \frac{d}{dt}\ln(\varsigma+f)=\frac{d\ln h}{dt} \quad\Rightarrow \frac{d}{dt}\ln\left(\frac{\varsigma+f}{h}\right)=0$$

$\frac{d}{dt}\ln\left(\frac{\varsigma+f}{h}\right)=0$ implies that $\frac{d}{dt}\left(\frac{\varsigma+f}{h}\right)=0$ BAROTROPIC VORTICITY EQUATION

$\eta\equiv\frac{\varsigma+f}{h}=$ potential vorticity $\Rightarrow$ potential vorticity is conserved following the motion in a barotropic atmosphere. This is also called '*Rossby potential vorticity*'.

For baroclinic fluids, potential vorticity is a function of all dependent variables of the fluid ($\vec{\mathbf{V}},\theta$ and ρ).

Conservation of potential vorticity is the air's equivalent of the conservation of angular momentum

POTENTIAL VORTICITY IN A BAROTROPIC FLUID

$$\frac{d}{dt}\left[\frac{\varsigma + f}{h}\right] \equiv 0$$

$\Rightarrow \dfrac{\varsigma + f}{h}$ is conserved following the fluid parcel in a barotropic fluid

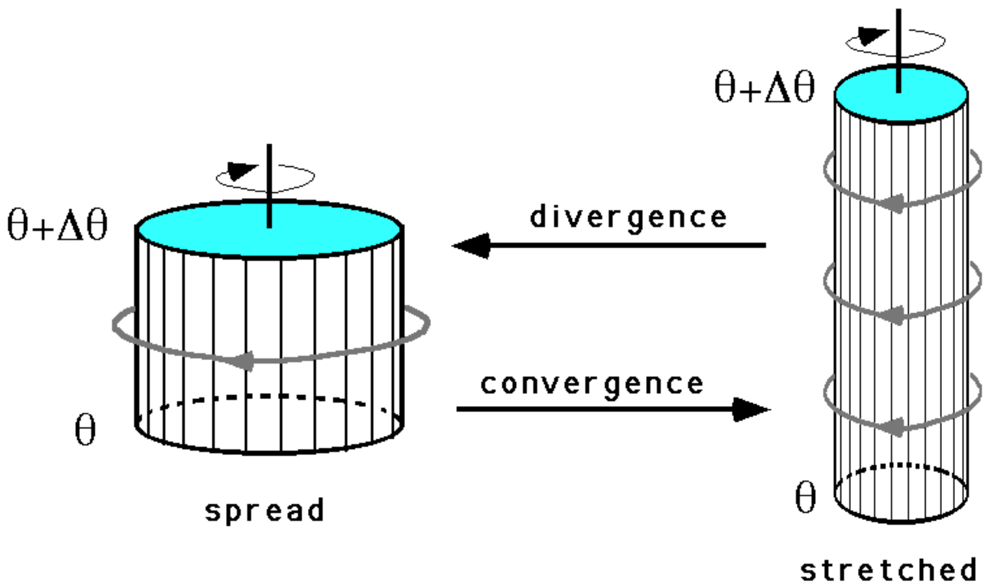
POTENTIAL VORTICITY IN A BAROCLINIC FLUID

On isentropic surfaces,

$$P = -g(\varsigma + f)\frac{\partial \theta}{\partial p} \quad (Ertel's\ potential\ vorticity)$$

P is conserved following an air parcel in adiabatic flow, and is therefore a good tracer of air parcels under conditions where diabatic heating (latent heat of condensation, radiation, etc.) can be neglected.

When a spinning ice skater has her arms spread out laterally, she spins slowly. When she contracts her arms, her rate of spin accelerates.



When air converges, the column stretches. To maintain potential vorticity, the air spins faster (vorticity increases), resulting in the stretched vortex on the right. Divergence, on the other hand, causes vortex spreading and slows down the rate of spin.

Generalized potential vorticity equation

The concept of potential vorticity (usually denoted by a variable q) was generalized by Ertel, during World War II, to include the complete 3D vorticity vector. Ertel's now-famous analysis, however was apparently done independently of Rossby's work.

Ertel's work included the effects of friction and diabatic heating, the potential vorticity equation was formulated with height as a vertical coordinate.

Compressible fluids

Equation of motion in 3D $\frac{\partial \vec{\mathbf{V}}}{\partial t} = -\left(\vec{\mathbf{V}} \cdot \nabla\right)\vec{\mathbf{V}} - f\hat{\mathbf{k}} \times \vec{\mathbf{V}} - \frac{1}{\rho}\nabla p - g\hat{\mathbf{k}} + \vec{\mathbf{F}}_{friction}$

Using the vector identity, $\left(\vec{\mathbf{A}} \cdot \nabla\right)\vec{\mathbf{A}} = \frac{1}{2}\nabla\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{A}}\right) - \vec{\mathbf{A}} \times \left(\nabla \times \vec{\mathbf{A}}\right)$, $\vec{\mathbf{A}} \equiv$ arbitrary vector, and let $\Phi \equiv gz$

We can write, $\frac{\partial \vec{\mathbf{V}}}{\partial t} = \vec{\mathbf{V}} \times \left(\nabla \times \vec{\mathbf{V}} + f\hat{\mathbf{k}}\right) - \nabla\left(\Phi + \frac{1}{2}\vec{\mathbf{V}} \cdot \vec{\mathbf{V}}\right) - \frac{1}{\rho}\nabla p + \vec{\mathbf{F}}_{friction}$ (1)

The differential form of thermodynamic energy equation is $Tds = c_p dT - \frac{dp}{\rho} \Leftrightarrow T\nabla s \cdot d\vec{\mathbf{r}} = c_p \nabla T \cdot d\vec{\mathbf{r}} - \frac{1}{\rho}\nabla p \cdot d\vec{\mathbf{r}}$

$s =$ specific entropy per unit mass $= c_p \ln\theta$; $d\vec{\mathbf{r}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$

It follows that $T\nabla s - c_p \nabla T = -\frac{1}{\rho}\nabla p$, substituting this in (1), we get

absolute vorticity $\nabla \times \vec{\mathbf{V}} + f\hat{\mathbf{k}} = \vec{\boldsymbol{\omega}}_{\mathbf{a}}$

$$\frac{\partial \vec{\mathbf{V}}}{\partial t} = \vec{\mathbf{V}} \times \left(\nabla \times \vec{\mathbf{V}} + f\hat{\mathbf{k}}\right) - \nabla\left(\Phi + c_p T + \frac{1}{2}\vec{\mathbf{V}} \cdot \vec{\mathbf{V}}\right) + T\nabla s + \vec{\mathbf{F}}_{friction}$$

$$\frac{\partial}{\partial t}\left[\nabla \times \vec{\mathbf{V}}\right] = \nabla \times \left[\vec{\mathbf{V}} \times \left(\nabla \times \vec{\mathbf{V}} + f\hat{\mathbf{k}}\right)\right] - \nabla \times \left[\nabla\left(\Phi + c_p T + \frac{1}{2}\vec{\mathbf{V}} \cdot \vec{\mathbf{V}}\right)\right] + \nabla \times T\nabla s + \nabla \times \vec{\mathbf{F}}_{friction}$$

READ Sec. 3.8 of Atmospheric dynamics – Mankin Mak (2011)

Generalized potential vorticity equation

$$\frac{\partial}{\partial t}\left[\nabla\times\vec{\mathbf{V}}\right]=\nabla\times\left[\vec{\mathbf{V}}\times\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\right]+\nabla\times T\nabla s+\nabla\times\vec{\mathbf{F}}_{friction}$$

Following the vector identities

$$\nabla\times\left(\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right)=\vec{\mathbf{A}}(\nabla\cdot\vec{\mathbf{B}})-\vec{\mathbf{B}}\left(\nabla\cdot\vec{\mathbf{A}}\right)-\left(\vec{\mathbf{A}}\cdot\nabla\right)\vec{\mathbf{B}}+\left(\vec{\mathbf{B}}\cdot\nabla\right)\vec{\mathbf{A}}$$

$$\nabla\left(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}\right)=\left(\vec{\mathbf{B}}\cdot\nabla\right)\vec{\mathbf{A}}+\left(\vec{\mathbf{A}}\cdot\nabla\right)\vec{\mathbf{B}}+\vec{\mathbf{B}}\times\left(\nabla\times\vec{\mathbf{A}}\right)+\vec{\mathbf{A}}\times\left(\nabla\times\vec{\mathbf{B}}\right)$$

$$\nabla\times\left(c\vec{\mathbf{A}}\right)=c\nabla\times\vec{\mathbf{A}}+\nabla c\times\vec{\mathbf{A}},\quad \nabla\cdot\nabla\times\vec{\mathbf{A}}=0,\ \nabla\times\nabla a=0$$

$$\frac{\partial\left(\nabla\times\vec{\mathbf{V}}\right)}{\partial t}=-\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\nabla\cdot\vec{\mathbf{V}}-\left(\vec{\mathbf{V}}\cdot\nabla\right)\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)+\left[\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right]\vec{\mathbf{V}}+\nabla T\times\nabla s+\nabla\times\vec{\mathbf{F}}_{friction}$$

$$\frac{d}{dt}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)=\left[\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right]\vec{\mathbf{V}}+\frac{1}{\rho}\frac{d\rho}{dt}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)+\nabla T\times\nabla s+\nabla\times\vec{\mathbf{F}}_{friction}$$

Dividing both sides by ρ

$$\frac{1}{\rho}\frac{d}{dt}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)-\frac{1}{\rho^2}\frac{d\rho}{dt}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)=\frac{1}{\rho}\left[\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right]\vec{\mathbf{V}}+\frac{1}{\rho}\nabla T\times\nabla s+\frac{1}{\rho}\nabla\times\vec{\mathbf{F}}_{friction}$$

Define the Ertel's potential vorticity by q, $\quad q=\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\nabla s$

$$\frac{dq}{dt}=\nabla s\cdot\frac{d}{dt}\left(\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\right)+\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\frac{d}{dt}\nabla s$$

$$\frac{d}{dt}\nabla s=\nabla\frac{\partial s}{\partial t}+\left(\vec{\mathbf{V}}\cdot\nabla\right)\nabla s$$

Generalized potential vorticity equation

$$\frac{1}{\rho}\frac{d}{dt}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)-\frac{1}{\rho^2}\frac{d\rho}{dt}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)=\frac{1}{\rho}\left[\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right]\vec{\mathbf{V}}+\frac{1}{\rho}\nabla T\times\nabla s+\frac{1}{\rho}\nabla\times\vec{\mathbf{F}}_{friction}\qquad (i)$$

Define the Ertel's potential vorticity by q, $\quad q=\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\nabla s$

$$\frac{dq}{dt}=\nabla s\cdot\frac{d}{dt}\left(\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\right)+\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\frac{d}{dt}\nabla s\qquad (ii)$$

$$\frac{d}{dt}\nabla s=\nabla\frac{\partial s}{\partial t}+\left(\vec{\mathbf{V}}\cdot\nabla\right)\nabla s$$

$$\left(\vec{\mathbf{V}}\cdot\nabla\right)\nabla s=\nabla\left(\vec{\mathbf{V}}\cdot\nabla s\right)-\left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}-\nabla s\times\left(\nabla\times\vec{\mathbf{V}}\right)$$

$$\frac{d}{dt}\nabla s=\nabla\frac{ds}{dt}+\left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}-\nabla s\times\left(\nabla\times\vec{\mathbf{V}}\right)\qquad (iii)$$

Substituting (iii), (i) in (ii)

$$\frac{dq}{dt}=\nabla s\cdot\left[\left(\frac{1}{\rho}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right)\vec{\mathbf{V}}+\frac{1}{\rho}\nabla T\times\nabla s+\frac{1}{\rho}\nabla\times\vec{\mathbf{F}}_{friction}\right]+\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\left(\nabla\frac{ds}{dt}+\left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}-\nabla s\times\left(\nabla\times\vec{\mathbf{V}}\right)\right)$$

Given that, $\nabla s\cdot\frac{1}{\rho}\nabla T\times\nabla s=0$ and using $\frac{1}{T}\dot{Q}=\frac{ds}{dt}=c_p\frac{d\ln\theta}{dt}$

$$\frac{dq}{dt}=\frac{1}{\rho}\left\{\nabla s\cdot\left[\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right]\vec{\mathbf{V}}+\nabla s\cdot\nabla\times\vec{\mathbf{F}}_{friction}\right\}+\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\nabla\left(\frac{\dot{Q}}{T}\right)-\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\ \left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}-$$

$$\frac{1}{\rho}f\hat{\mathbf{k}}\cdot\nabla s\times\left(\nabla\times\vec{\mathbf{V}}\right)$$

Generalized potential vorticity equation

$$\frac{dq}{dt}=\frac{1}{\rho}\left\{\nabla s\cdot\left[\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\right]\vec{\mathbf{V}}+\nabla s\cdot\nabla\times\vec{\mathbf{F}}_{friction}\right\}+\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\nabla\left(\frac{\dot{Q}}{T}\right)-\frac{\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}}{\rho}\cdot\left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}-\frac{1}{\rho}f\hat{\mathbf{k}}\cdot\nabla s\times\left(\nabla\times\vec{\mathbf{V}}\right)$$

It is a vector property that $\nabla s\cdot\left(\nabla\times\vec{\mathbf{V}}\cdot\nabla\right)\vec{\mathbf{V}}=\left(\nabla\times\vec{\mathbf{V}}\right)\cdot\left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}$

It is left as an exercise for the reader to verify that $\nabla s\cdot\left(f\hat{\mathbf{k}}\cdot\nabla\right)\vec{\mathbf{V}}-f\hat{\mathbf{k}}\cdot\left(\nabla s\cdot\nabla\right)\vec{\mathbf{V}}-f\hat{\mathbf{k}}\cdot\nabla s\times\left(\nabla\times\vec{\mathbf{V}}\right)=0$

Upon substitution,

$$\frac{dq}{dt}=\frac{1}{\rho}\nabla s\cdot\nabla\times\vec{\mathbf{F}}_{friction}+\frac{1}{\rho}\left(\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}\right)\cdot\nabla\left(\frac{\dot{Q}}{T}\right)=\frac{1}{\rho}\nabla s\cdot\nabla\times\vec{\mathbf{F}}_{friction}+\frac{\vec{\boldsymbol{\omega}}_a}{\rho}\cdot\nabla\left(\frac{\dot{Q}}{T}\right)$$

For frictionless, $\vec{\mathbf{F}}_{friction}\equiv 0$, and adiabatic $\dot{Q}\equiv 0 \quad\Rightarrow \frac{dq}{dt}=0$ (Rossby's potential vorticity equation)

The derivation is valid even if the atmosphere is non-hydrostatic as the derivation did not make use of hydrostatic assumption. Thus, the equation for Ertel potential vorticity (q) has broad applications for flow in the atmosphere.

absolute vorticity $\nabla\times\vec{\mathbf{V}}+f\hat{\mathbf{k}}=\vec{\boldsymbol{\omega}}_{\mathbf{a}}$

q of a fluid parcel would increase if

(i) the gradient of heating has component in the direction of absolute vorticity vector $\vec{\boldsymbol{\omega}}_a$

(ii) the curl of the frictional force has a component in the direction of the gradient of potential temperature.

Rossby, C. G., (1940): Planetary flow patterns in the atmosphere. Quart. J. Roy. Met. Soc., 66, 68-87.

Ertel, H., (1942): Ein Neuer hydrodynamischer Wirbelsatz. Met. Z., 271-281.

PV of a compressible fluid

$$\frac{dq}{dt} = \frac{\vec{\boldsymbol{\omega}}_a}{\rho} \cdot \nabla\left(\frac{\dot{Q}}{T}\right) + \nabla s \cdot \frac{\nabla \times \vec{\mathbf{F}}_{friction}}{\rho}$$

$s = c_p \ln\theta =$ specific entropy per unit mass

$q =$ Ertel potential vorticity of a compressible fluid

$$\frac{dq}{dt} \equiv \frac{d}{dt}\left(\frac{\vec{\boldsymbol{\omega}}_a \cdot \nabla\theta}{\rho}\right) = \frac{\vec{\boldsymbol{\omega}}_a \cdot \nabla\dot{Q}}{\rho} + \frac{\nabla\theta \cdot \left(\nabla \times \vec{\mathbf{F}}_{friction}\right)}{\rho}$$

$$q = \frac{\vec{\boldsymbol{\omega}}_a \cdot \nabla\theta}{\rho} \approx \frac{\omega_z}{\rho}\frac{\partial\theta}{\partial z} = \frac{\zeta + f}{\rho}\frac{\partial\theta}{\partial z}$$

$$\Leftrightarrow \quad q = -g(\zeta + f)_\theta \frac{\partial\theta}{\partial p} = \text{constant.}$$

q is often referred to as the IPV

The PV of a fluid parcel would increase if

(a) the gradient of heating has a component in the direction of the absolute vorticity vector and/or

(b) the curl of thefrictional force has a component in the direction of the gradient of θ.

q is conserved under adiabatic and inviscid conditions

Why vertical component is important for synoptic scale motions?

$$\frac{dq}{dt} = \frac{\vec{\boldsymbol{\omega}}_a}{\rho} \cdot \nabla \left(\frac{\dot{Q}}{T} \right) + \nabla s \cdot \frac{\nabla \times \vec{\mathbf{F}}_{friction}}{\rho}$$

$s = c_p \ln \theta =$ specific entropy per unit mass.

$q \equiv \dfrac{\vec{\boldsymbol{\omega}}_a \cdot \nabla s}{\rho} =$ Ertel potential vorticity of a compressible fluid,

q is conserved under adiabatic and inviscid conditions.

For large-scale atmospheric motions,

	Vertical	Horizontal
$\lvert\vec{\boldsymbol{\omega}}_a\rvert$	$10^{-4} - 10^{-5}\ \mathrm{s}^{-1} \left(\approx f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$	$10^{-3}\ \mathrm{s}^{-1} \left(\approx \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right)$
$\left\lvert \frac{\nabla \theta}{\theta} \right\rvert = \nabla s$	$10^{-5} - 10^{-2}\ \mathrm{s}^{-1}$	$10^{-8}\ \mathrm{s}^{-1}$

This suggests that the vertical component of $\vec{\omega}_a$ is rather important for large-scale (synoptic scale) motions.

Therefore, a simplified Ertel's formula can be written as follows:

$$q = \frac{\vec{\boldsymbol{\omega}}_a \cdot \nabla \theta}{\rho} \approx \frac{\omega_z}{\rho} \frac{\partial \theta}{\partial z} = \frac{\zeta + f}{\rho} \frac{\partial \theta}{\partial z} \quad \Leftrightarrow \quad q = -g(\zeta + f) \frac{\partial \theta}{\partial p} = \text{constant}.$$

q is often referred to as the "Isentropic Potential Vorticity (IPV)"

The potential vorticity is a quantity that is related to the absolute vorticity ($\vec{\omega}_a$) and the stratification ($\nabla\theta$) that is materially conserved in the absence of friction or diabatic heating.

Use of potential vorticity

The real atmosphere is usually baroclinic, so it is more appropriate to use conservation of Ertel's potential vorticity.

$$q = -g(\varsigma + f)\frac{\partial\theta}{\partial p} = -g\zeta_{a\theta}\frac{\partial\theta}{\partial p} \quad \Rightarrow \ \zeta_{a\theta} = \text{ component of absolute vorticity}$$

normal to an isentropic surface. P is also referred to as the ísentropic potential vorticity (IPV)

If the flow is adiabatic, any change in IPV must be due to the advection of IPV.
If the flow is not adiabatic, IPV can be used to diagnose where and when the diabatic processes are acting to influence the flow.

Since diabatic processes are associated with creation and destruction of PV, the Lagrangian rate of change of PV is

$$\frac{dq}{dt} \approx -g\left(\varsigma + f\right)\frac{\partial}{\partial p}\left(\frac{d\theta}{dt}\right) \Leftrightarrow \text{ PV is increased when the vertical gradient of diabatic heating is}$$

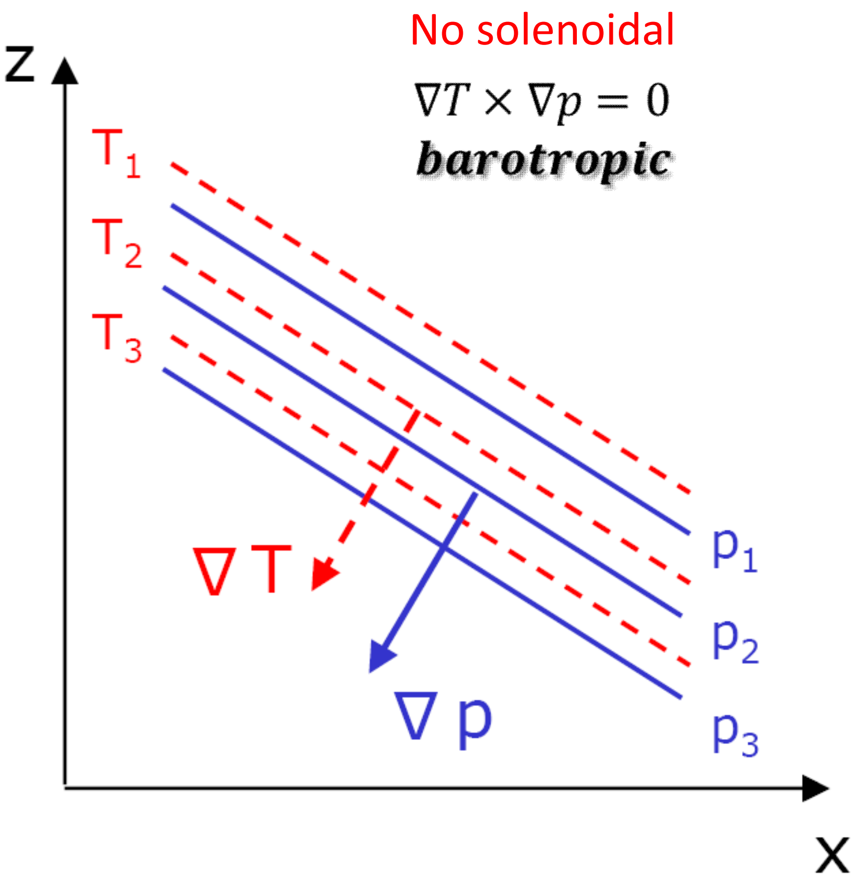
positive.

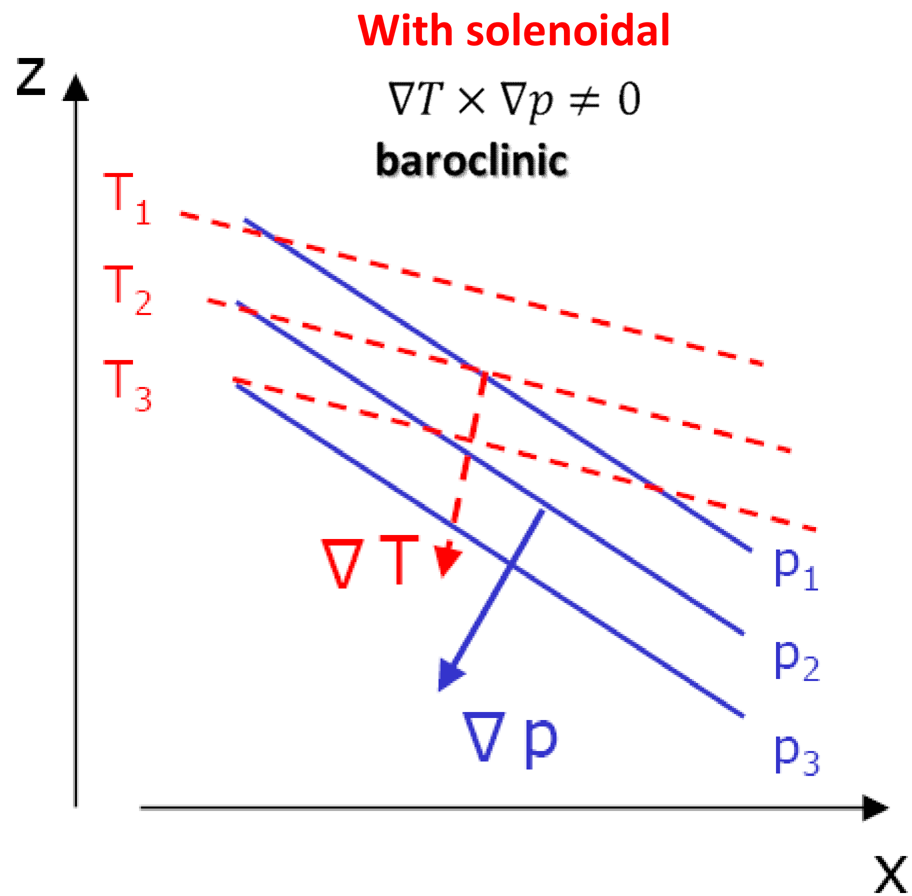
What is barotropic or baroclinic?

- *The atmosphere is both barotropic and part baroclinic. Barotropic is very consistent, no air masses, no fronts and is characteristic of the "tropics". Baroclinic is much more variable. Different air masses, cold fronts, development of cyclones. Baroclinic is characteristic of extra-tropical regions.*

- ***BAROTROPIC***
  - ***Region of uniform temperature distribution; A lack of fronts. Everyday being about hot and humid (with no cold fronts to cool things off) surroundings would be a barotropic type atmosphere. Part of the word barotropic is tropic. The tropical latitudes are barotropic as there are no fronts in the tropics.***
- ***BAROCLINIC***
  - ***Distinct air mass regions exist. Fronts separate warmer from colder air. In a synoptic scale baroclinic environment you will find the polar jet in the vicinity, troughs of low pressure (mid-latitude cyclones) and frontal boundaries.***
  - ***There are clear density gradients in a baroclinic environment caused by the fronts. Any time you are near a mid-latitude cyclone you are in a baroclinic environment.***
  - ***Part of the word baroclinic is clinic. If the atmosphere is out of balance, it is baroclinic, just as if a person felt out of balance they would need to go to a clinic.***

Barotropic and baroclinic

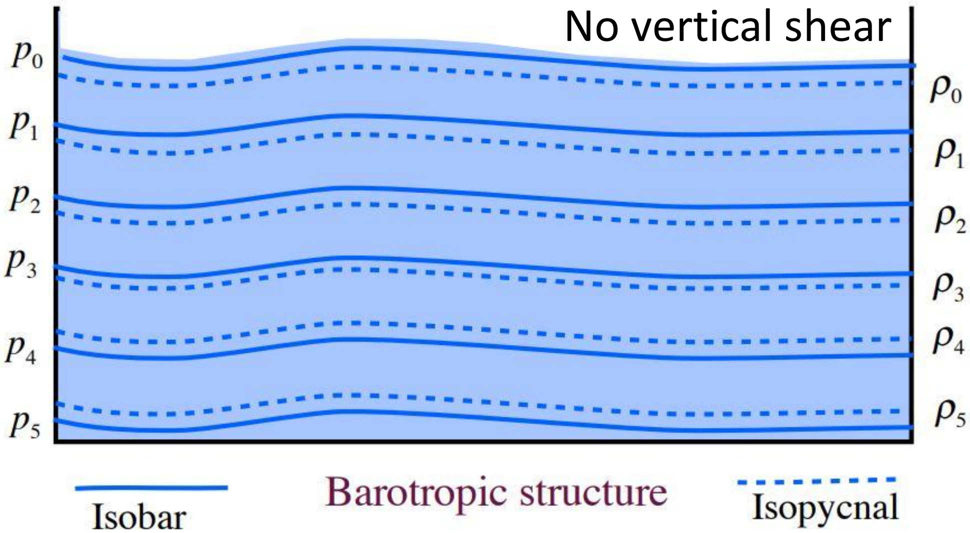


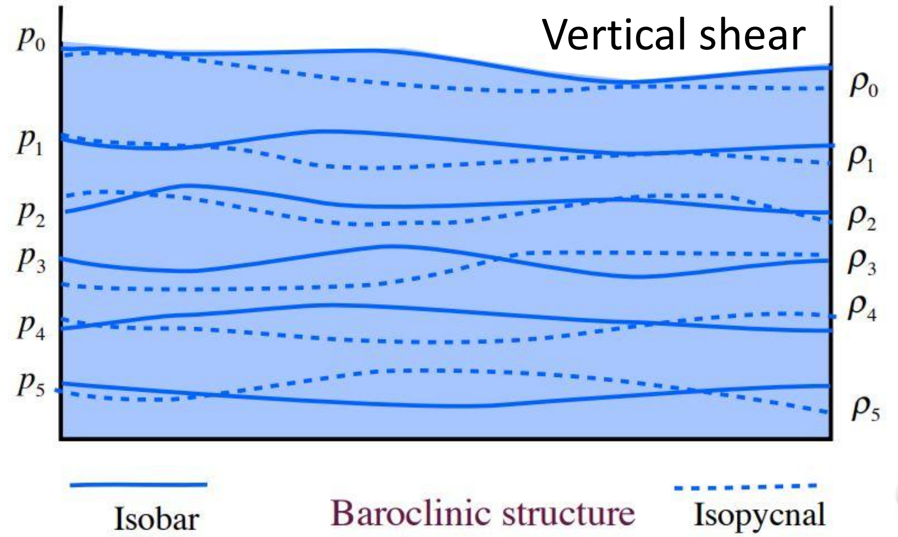


A barotropic atmosphere is one in which the density depends only in the pressure, so that isobaric surfaces are also surfaces of constant density.

An atmosphere in which density depends on both temperature and pressure is called an baroclinic atmosphere.

Idealized barotropic and baroclinic structures





Temperatures in a barotropic environment are homogenous and uniform. Warm and humid throughout the year with little fluctuation in temperature.

In a barotropic environment in the tropics there is ***NO*** *advections such as cold air advection or warm air advection,*

- *No fronts such as cold or warm fronts, nor any occluded fronts are present here.*

Baroclinic motions :

Temperatures in a baroclinic environment are **heterogenous** – which means large temperature differentials.

Lower and upper troposphere communicates with a finite vertical velocity.

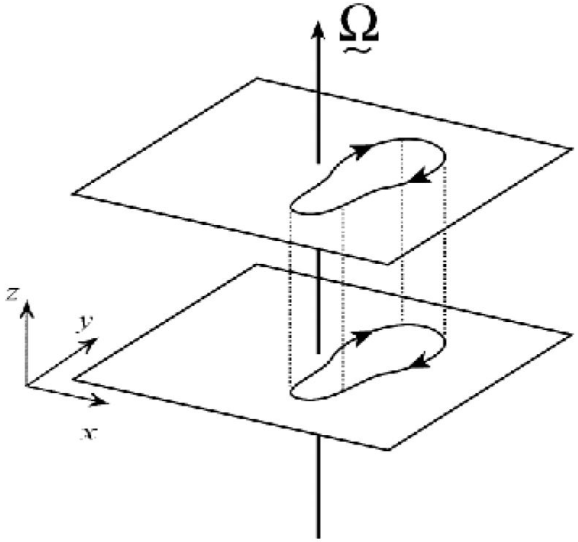
In a Baroclinic environment the upper levels (500 mb and above) are characterized by large WAVES – Ridges and Troughs. Primarily the Polar Jetstream governs the troughs and Ridges

Taylor-Proudman Theorem

If the fluid is homogeneous (ρ uniform) then, the geostrophic flow is two-dimensional and does not vary in the direction of the rotation vector, $\vec{\mathbf{\Omega}}$.

$$\left(u_g, v_g\right) = \frac{1}{f\rho}\left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}\right) \Rightarrow$$

If ρ and f are constant, then taking the vertical derivative of the geostrophic flow components and using hydrostatic balance, we see that $\left(\frac{\partial u_g}{\partial z}, \frac{\partial v_g}{\partial z}\right) = 0$



If the flow is sufficiently slow and steady (Ro << 1) and friction is negligible, then $2\mathbf{\Omega} \times \vec{\mathbf{V}} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{k}} = 0$

Taking the curl $\left[\nabla \times \left\{2\mathbf{\Omega} \times \vec{\mathbf{V}} + \frac{1}{\rho}\nabla p + g\hat{\mathbf{k}}\right\}\right]$, we find that

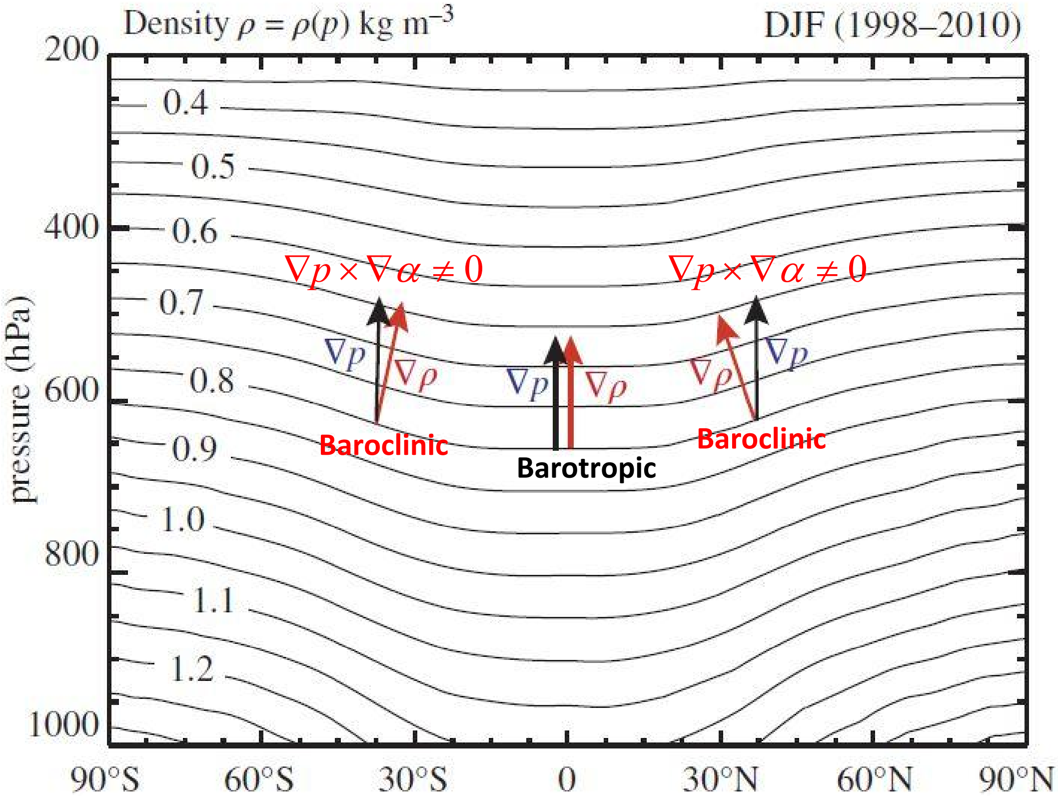
if the fluid is barotropic [i.e., one in which $\rho = \rho(p)$, then

$(\mathbf{\Omega} \cdot \nabla)\vec{\mathbf{V}} = 0.$

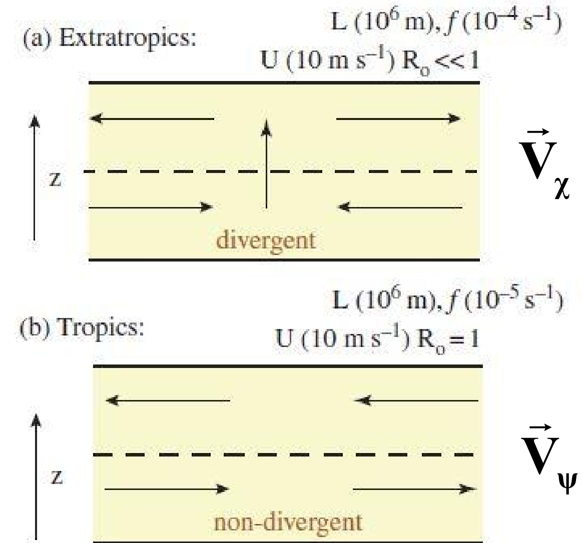
Since $\mathbf{\Omega} \cdot \nabla$ is the gradient operation in the direction of $\mathbf{\Omega}$, i.e., $\hat{\mathbf{k}}$

$$\Leftrightarrow \frac{\partial \vec{\mathbf{V}}}{\partial z} = 0 \Leftrightarrow \text{ barotropic = no vertical shear}$$

The Taylor-Proudman theorem, states that slow, steady, frictionless flow of a barotropic, incompressible fluid is two-dimensional and does not vary in the direction of the rotation vector $\mathbf{\Omega}$



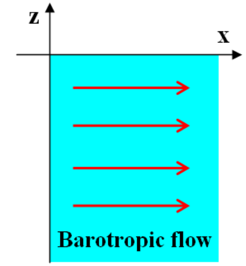
Baroclinic motions : lower and upper troposphere communicates with a finite vertical velocity.

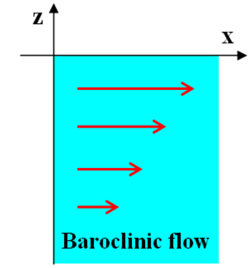


Cross-section of the zonally averaged isopycnals plotted against pressure for the NH winter (1998.2010: December.February).

→ In the middle latitudes, in the vicinity of the westerly maximum, the atmosphere is highly baroclinic, as indicated by the large angle between the ρ and p gradients (vectors ∇p and $\nabla \rho$).

⇒ In the tropics, however, the gradients are almost parallel, indicating a lack of baroclinicity. The root word of Barotropic is tropic.





Baroclinic vs. Barotropic

Barotropic	Baroclinic
ρ=ρ(p) only **Implications:** 1) isobaric and isothermal surfaces coincide $\nabla p \times \nabla \alpha = 0$ 2) no vertical wind shear (thermal wind = 0) 3) no tilt of pressure systems with height	ρ=ρ(p,T) **Implications:** 1) isobaric and isothermal surfaces intersect $\nabla p \times \nabla \alpha \neq 0$ 2) vertical wind shear (thermal wind ≠ 0) 3) pressure systems tilt with height

Seasons: Atmosphere is most baroclinic in winter.
Atmosphere is least baroclinic in summer.

Geographic: Atmosphere is most baroclinic in midlatitudes
Atmosphere is least baroclinic in the Tropics

Equivalent Barotropic System

- *Barotropic systems are characterized by a lack of wind shear (temperature is uniform, no temperature gradient). Usually, in operational meteorology, references to barotropic systems refer to equivalent barotropic systems - systems in which temperature gradients exist, but are parallel to height gradients on a constant pressure surface. In such systems, height contours and isotherms are parallel everywhere, and winds do not change direction with height.*

  While some systems (such as closed lows or cutoff lows) may reach a state that is close to equivalent barotropic, the term barotropic system usually is used in a relative sense to describe systems that are really only close to being equivalent barotropic, i.e., isotherms and height contours are nearly parallel everywhere and directional wind shear is weak.

What is equivalent barotropic?

In the atmosphere the isotherms are sometimes parallel to the height contours. Often times meteorologists say "barotropic" when they really mean "equivalent barotropic"

- *If the isotherms are very widely spaced then the region or level is close to barotropic.*
- *If the isotherms are parallel to the height contours then the region or level is equivalent barotropic.*
- *If the isotherms cross the height contours the region or level is baroclinic*

Variation of wind with height is vertically averaged assuming that the thermal wind is in the same direction as the geostrophic wind at all heights

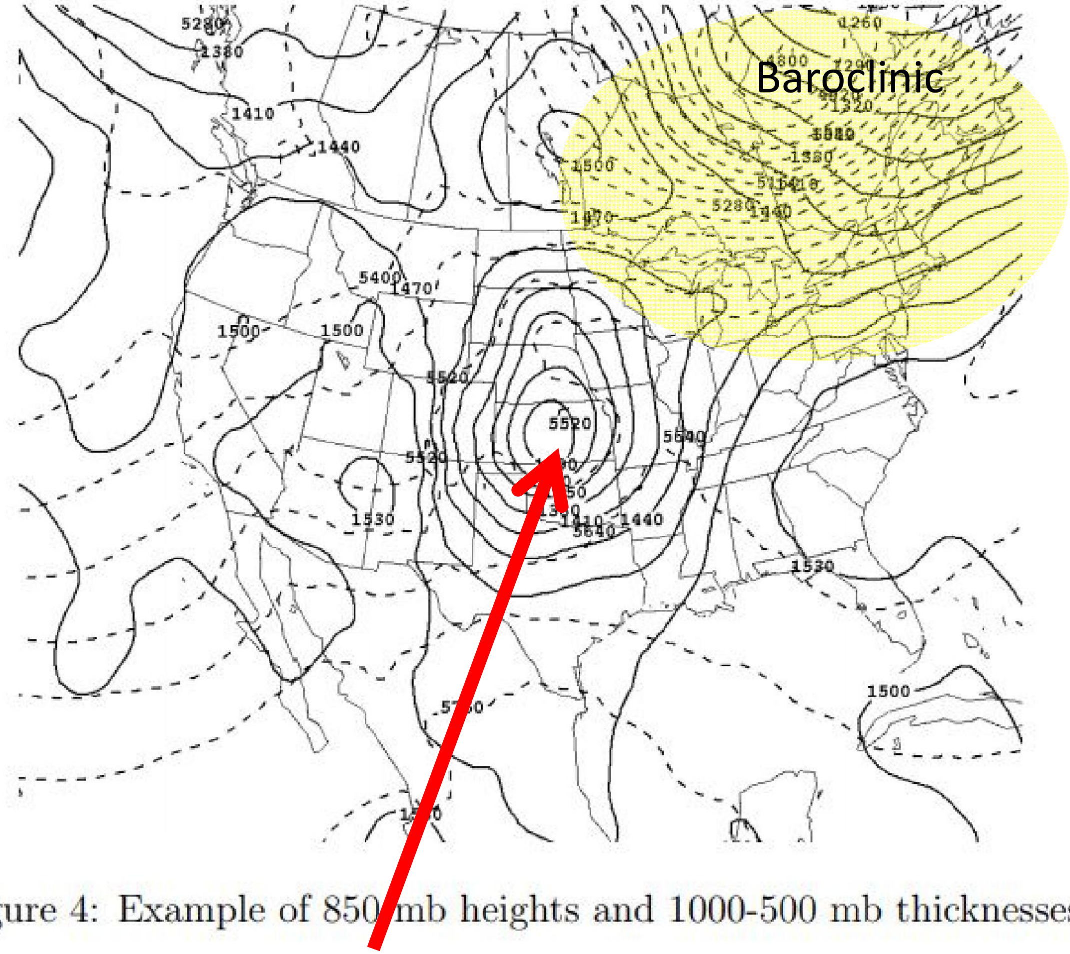
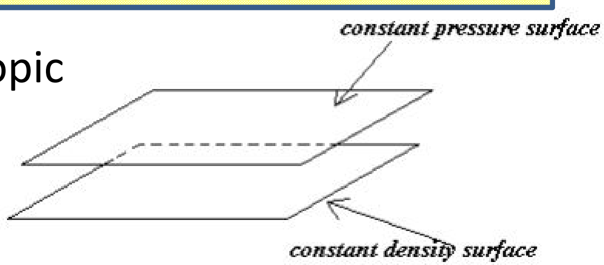
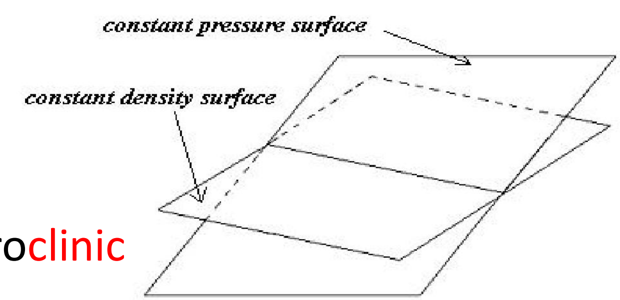


Figure 4: Example of 850 mb heights and 1000-500 mb thicknesses.

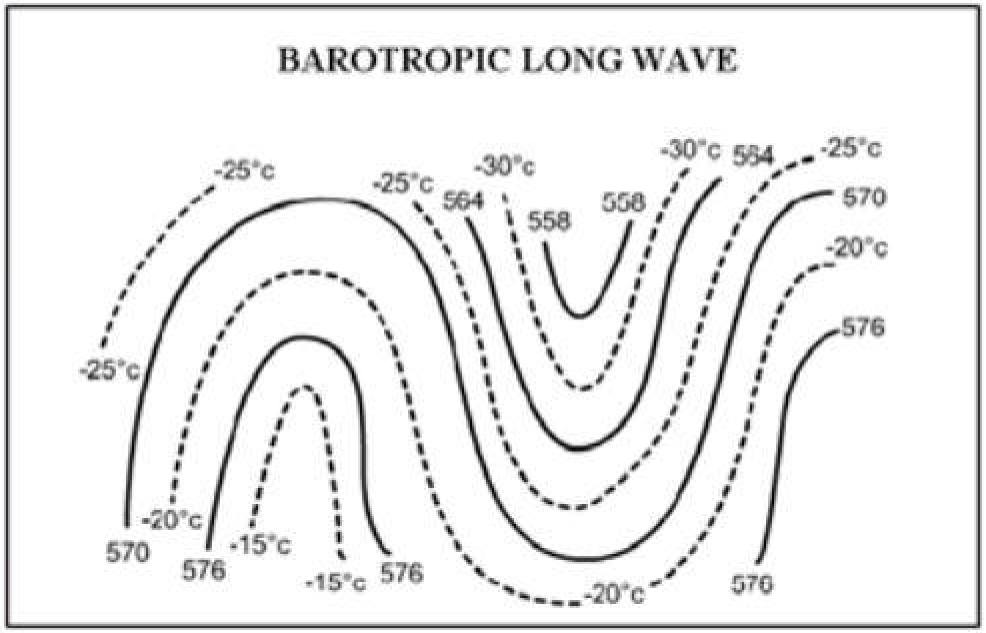
The low over Kansas at 850 mb is nearly equivalent barotropic since the height and thickness lines are nearly parallel

Barotropic





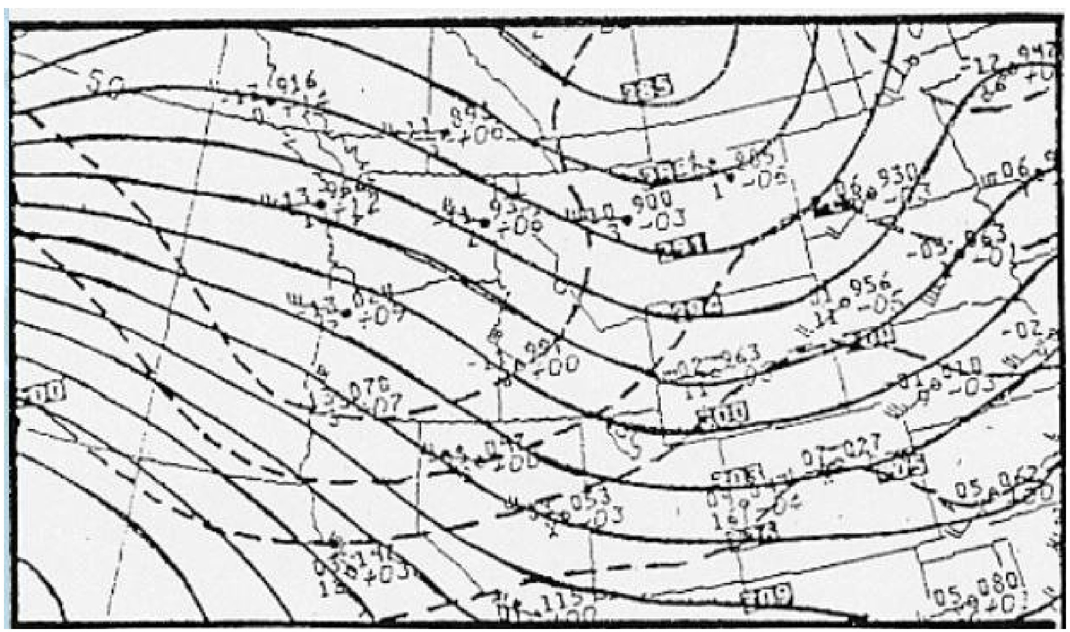
Baroclinic



Equivalent barotropic:

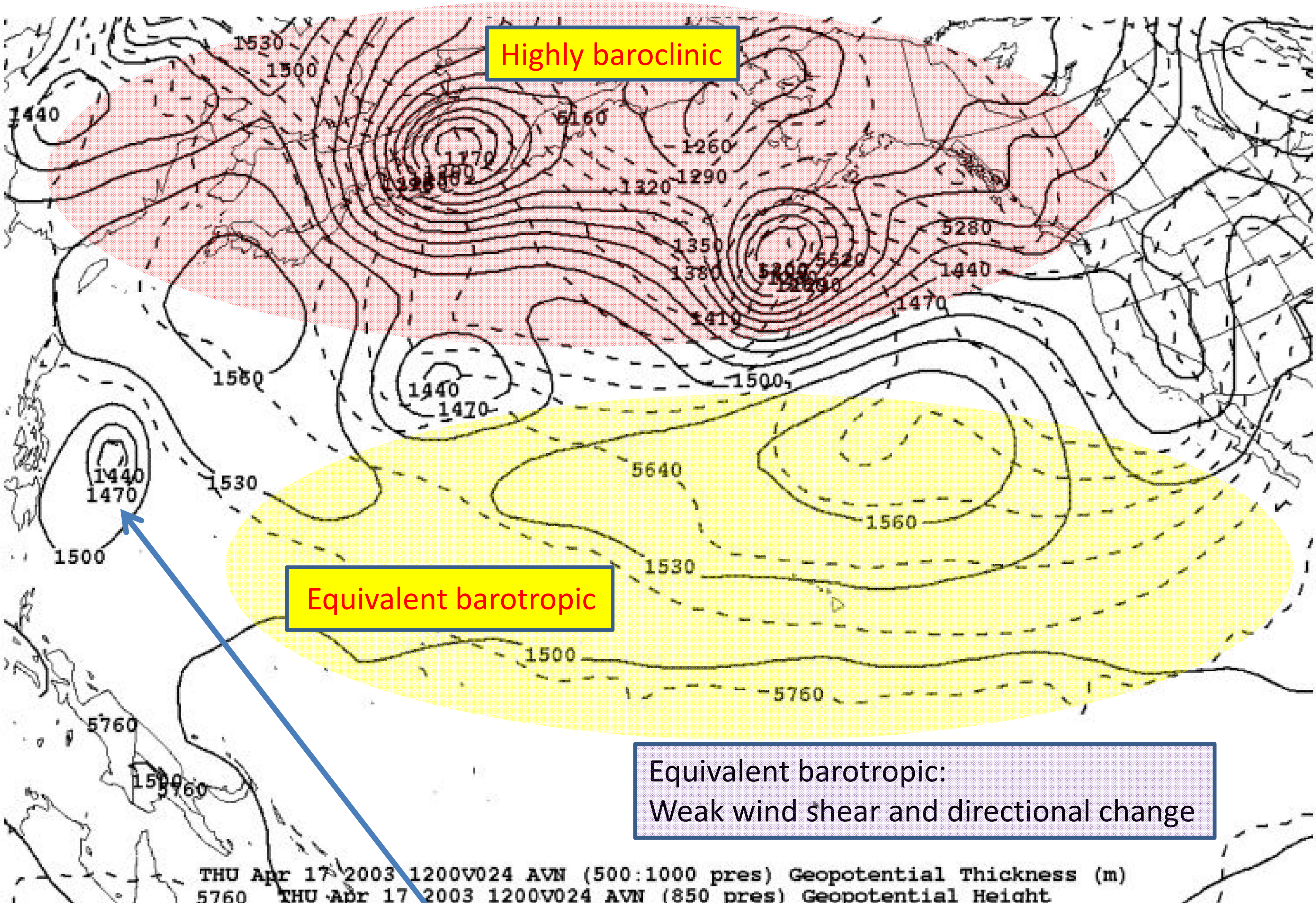
- ***Thermal/contour trough axes in phase.***
- ***Thermal/contour ridge axes in phase.***
- ***Longwave troughs - cold core***
- ***Longwave ridges - warm core.***

Baroclinic:



The state of the atmosphere where isotherms exist on isobaric charts and these isotherms intersect the height contours (i.e., isotherms and height contours are "out-of-phase" with one another).

Vertical shear is allowed. Wind direction changes with height, and is usually accompanied by speed changes.



Tropics: height contours and thickness contours are spread very far apart

Note that equivalent barotropic low near the Philippines.

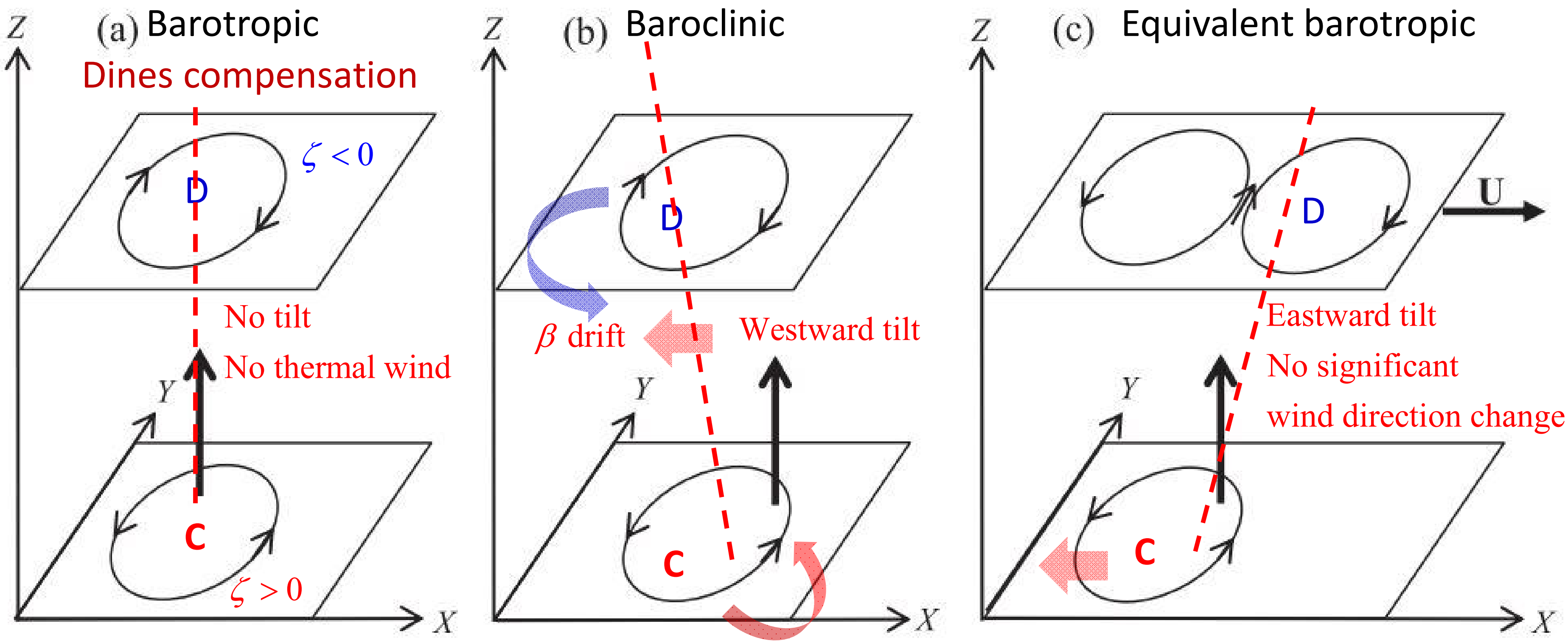


FIG. 12. Schematic diagram of the formation of baroclinic and barotropic structures in the NH tropics. (a) Deep convective heating is balanced by adiabatic cooling associated with ascent. Divergence above the maximum ascent gives an anticyclonic vorticity tendency there, and convergence below gives a cyclonic vorticity tendency there. (b) The β effect leads to a westward drift both above the heating maximum and below it. A steady situation is reached when the upper equatorward motion and the lower poleward motion are in the longitude of the ascent, giving a baroclinic structure in the vertical. (c) In the presence of strong westerly shear and upper-tropospheric westerly winds relative to the motion of the convection, zonal advection can dominate the β effect there and lead to eastward displacement of the upper wave. If the β effect still dominates for the lower wave, the resulting structure is equivalent barotropic.

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The Equivalent Barotropic Structure of Waves in the Tropical Atmosphere in the Western Hemisphere

GUI-YING YANG
National Centre for Atmospheric Science, and University of Reading, Reading, United Kingdom

BRIAN J. HOSKINS
University of Reading, Reading, and Grantham Institute for Climate Change, Imperial College, London, United Kingdom

Summary: Vorticity and Circulation

Vorticity is defined to be the curl of velocity, and normally denoted by the symbol $\vec{\boldsymbol{\omega}}$

The three-dimensional vorticity vector is given by:

$$\vec{\boldsymbol{\omega}} \equiv \nabla \times \vec{\mathbf{V}} \equiv \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \equiv \hat{\mathbf{i}}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In atmospheric and oceanic sciences, we are primarily concerned with circulations in the horizontal plane, vorticity implies the vertical component (unless otherwise stated.)

$$\zeta = \hat{\mathbf{k}} \cdot \vec{\boldsymbol{\omega}} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$ (vorticity has the units analogous to "rotations" per second).

Circulation is defined to be the integral of velocity around a closed loop $C = \oint \vec{\mathbf{V}} \cdot d\mathbf{l}$

The circulation round any reducible closed curve is equal to the integral of vorticity over an open surface bounded by the curve and, equivalently, is equal to the strength of the vortex-tube formed by all the vortex lines passing through the curve.

$$C = \oint_{line} \vec{\mathbf{V}} \cdot d\mathbf{l} = \int_{Area} \left(\nabla \times \vec{\mathbf{V}}\right) \cdot \hat{\mathbf{n}}\, d\vec{\mathbf{A}}$$

Circulation Theorems

- Bjerknes: Absolute circulation is changed by solenoidal term
- Kelvin: Absolute circulation is conserved in barotropic fluids

$$\frac{dC}{dt} = \frac{d}{dt}\oint \vec{\mathbf{V}} \cdot d\mathbf{l}$$

In a barotropic fluid, $\rho = \rho(p)$ therefore an exact differential $\Leftrightarrow$ Kelvin's theorem

In a baroclinic fluid, $\rho = \rho(p,T)$ Not exact differential $\Leftrightarrow$ Bjerknes theorem

Kelvins circulation theorem

$$\oint \alpha dp = \oint f(p)dp = 0 \Leftrightarrow \frac{dC}{dt} = 0$$

$$\oint \frac{dp}{\rho(p)} = \frac{1}{\rho(p_2)} - \frac{1}{\rho(p_1)} = 0$$

circulation conservation

$$C_{abs} = C_{rel} + (2\Omega\sin\varphi)A = \text{ constant}$$

states that the circulation around a closed curve moving with a frictionless, barotropic fluid (no solenoids) is constant $\Leftrightarrow \zeta + f \equiv$ conserved, gives rise to Rossby waves in barotropic fluids

$\Rightarrow$ If baroclinicity arises, solenoid brings the conditions back to barotropic through a closed path of circulation

Bjerknes circulation theorem

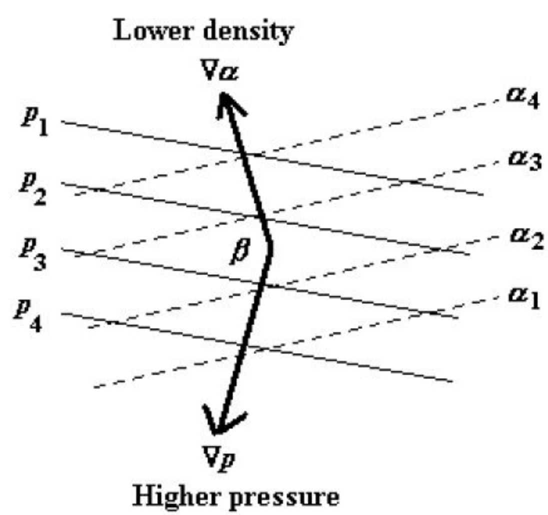
In a baroclinic fluid $\oint \frac{dp}{\rho} = -\oint RT \ln p$

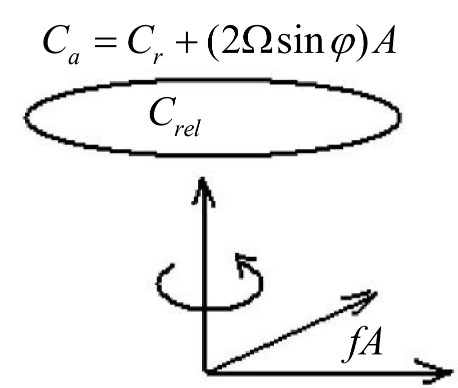
$$\frac{dC}{dt} = \iint_A (\nabla p \times \nabla \alpha) \cdot d\mathbf{A} = \iint_A |\nabla \alpha||\nabla p| \sin\beta \, dA$$

$$\frac{dC}{dt} = -R\iint_A (\nabla T \times \nabla \ln p) \cdot d\mathbf{A}$$

Change of circulation is by baroclinicity

$(\nabla p \times \nabla \alpha) \equiv$ solenoidal term





THE BJERKNES' CIRCULATION THEOREM

A Historical Perspective

BY ALAN J. THORPE, HANS VOLKERT, AND MICHAŁ J. ZIEMIAŃSKI

Other physicists had already made the mathematical extension of Kelvin's theorem to compressible fluids, but it was not until Vilhelm Bjerknes' landmark 1898 paper that meteorology and oceanography began to adopt this insight.

Bulletin of American Meteorological Society (2003), Pages 471-480