Chapter-II

PERTURBATION THEORY

Main goal in Meteorology is to forecast the weather parameters for the future time with the knowledge of their present value. Bjerkness (1904) had recognized this problem of weather forecasting as an initial value problem(IVP).

Initial value problem is a partial differential equation (Linear/ Non-Linear) with time (t) as an in dependent variable.

Some Useful Concepts :

Partial derivative:

Let a quantity 'S' is dependent on x, y, z, t. Then derivative of S with respect to any one (say t) of these four, keeping rest three unchanged, is called partial derivative of S with respect to 't'. For example 24 hrs change of pressure at a place is the partial change in pressure with respect to time. These are denoted by $\frac{\partial s}{\partial t}, \frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial s}{\partial z}$ etc.

Examples: Let, $V = x^3 + y^3 + 3axyz$

Hence, $\frac{\partial V}{\partial x} = 3x^2 + 3ayz$ (y, z have been kept constant)

$\frac{\partial V}{\partial y} = 3y^2 + 3axz$ (z, x have been kept constant)

$\frac{\partial V}{\partial z} = 3axy$ (x, y have been kept constant)

Partial differential equation (PDE):

A differential equation is an equation which involves derivative or differential of the dependent variable. A PDE is an equation which involves partial derivatives or differentials of the dependent variable.

EX: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$ is a partial differential equation, as it contains the partial derivatives of the dependent variables u, p.

Order of a PDE :

It is the highest order partial derivatives involved in the equation.

Ex. Consider the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y)$.

Here, u is the dependent variable, x, y are independent variables and F(x ,y) is a known function of x,y. In the PDE the highest order partial derivative involved in this equation is 2. So the order of this PDE is 2.

Linear and non-linear PDE :

A general form of a 2nd order PDE is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G .$$

In the above equation A,B,C,D,E,F and G are called coefficients of the PDE. If all these coefficients are constants or functions of independent variables (x , y), then the resulting PDE is known as a Linear PDE.

For example let us consider the following PDE:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

For this PDE A = 1

B = 2

C = 1 and

D = E = F = G=0. Hence this PDE is a Linear PDE.

We consider another PDE,

$y^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = (x + y)$. In this PDE, A, B, C and G are functions of x or y or both. So, this is also a 2nd order linear PDE.

on the other hand if at least one these coefficients is a function dependent variable, then the resulting PDE is known as a non-linear PDE.

For example let us consider the following PDE:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} .$$

In the above equation, A = B = C = F = 0, D = u and E = v. Since u, v are dependent variables, hence it is a non-linear PDE.

Need for the perturbation theory :

There are several method for weather forecasting, viz. synoptic, statistical, Dynamic (Numerical weather prediction) method etc.

In the NWP, the governing equations are solved for the weather parameter, viz. u,v,w, T, P etc.. The governing equations are non-linear partial differential equation. Non-linear partial differential equations can not be solved exactly, as till now we don't have any method to get exact solution of non-linear partial differential equation.

To get rid of the above problem, there are two ways viz.,

(a) Transform the non-linear partial differential equation into ordinary differential equation and then get exact solution.

(b) Transform the set of partial differential equations into their finite difference form and then solve them numerically.

Discussion about (a) is beyond the scope of discussion. Now while discussing (b), it is worth mentioning that the numerical solution of these non-linear partial differential equation is highly sensitive to the initial conditions given, i.e. a slight change in the initial condition may lead to an abrupt change in the numerical solution. This is due to the presence of non-linearity in the governing equations. Perturbation theory was proposed to remove the non-linearity from the governing equations.

Basic postulates of perturbation theory :

This theory is based on same postulates, which are given below :

I. According to this theory, the total atmospheric flow consists of a mean flow and a perturbation superimposed on it. So, that all field variables consist of a basic (mean) part and a perturbation part.

II. Both the mean part and the total (mean + perturbation) satisfy the governing equations. Mean part is the temporal and longitudinal average of the variable as a result of which it is independent of x and t.

III. The magnitude of perturbation part is very small as compared to that of mean part, so that any product of perturbations or product of their derivatives or product of a perturbation and derivative of perturbation may be neglected.

Now, it is our task, to verify whether using the above postulates, the non-linearity from a

term of governing equation may be removed or not.

For that we consider an arbitrary non-linear term, say, $u\dfrac{\partial \varphi}{\partial x}$.

Using postulate (I), $u = \overline{u} + u'$ and $\varphi = \overline{\varphi} + \varphi'$.

Hence, $u\dfrac{\partial \varphi}{\partial x} = (\overline{u} + u')\dfrac{\partial(\overline{\varphi} + \varphi')}{\partial x} = (\overline{u} + u')\left(\dfrac{\partial \overline{\varphi}}{\partial x} + \dfrac{\partial \varphi'}{\partial x}\right) = \overline{u}\dfrac{\partial \varphi'}{\partial x} + u'\dfrac{\partial \varphi'}{\partial x}$ (Here, $\dfrac{\partial \overline{\varphi}}{\partial x} = 0$, as per 2nd part of postulate (II)). Again using postulate (III), $u'\dfrac{\partial \varphi'}{\partial x} \approx 0$, being a product of perturbation quantity and its derivative.

Hence using perturbation technique, non-linearity from the governing equations may be removed.