IMD-WMO Joint group fellowship training program on NWP

Theme: Numerical methods

Lecture-3-8

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Different types of differential eqations

- Categorization –I:
  - Ordinary: $\frac{d^2u}{dt^2} + \omega^2 u = 0$
  - Partial: $\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - \alpha\frac{\partial p}{\partial z} - g$; $\frac{\partial^2 T}{\partial t^2} = -k\frac{\partial^2 T}{\partial x^2}$
- Categorization – II:
  - Linear: $\frac{\partial w}{\partial t} = -c\frac{\partial w}{\partial x}; c\ is\ constant$
  - Non-Linear: $\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$

General form of 2nd order PDE

General form of 2nd order PDE is: $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$

A,B,C,D,E,F are coefficients, may be all of them constants or a couple/all of them may be functions of x,y; G is a known quantity may be a constant or a function of x,y and u(x,y) is an unknown function to be determined.

If all these coefficients are constants or functions of independent variables (x , y), then the resulting PDE is known as a Linear PDE. Example: $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

On the other hand, if at least one these coefficients is a function dependent variable, then the resulting PDE is known as a non-linear PDE. Example: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

Governing equations of NWP

- $\frac{\partial u}{\partial t} = -\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]u - \frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega(w\cos\varphi - v\sin\varphi) + \frac{uv}{a}tan\varphi - \frac{uw}{a} + \frac{\mu}{\rho}\nabla^2 u,$
- $\frac{\partial v}{\partial t} = -\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]v - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega(u\sin\varphi) - \frac{u^2}{a}tan\varphi - \frac{vw}{a} + \frac{\mu}{\rho}\nabla^2 v$
- $\frac{\partial w}{\partial t} = -\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]w - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega(u\cos\varphi) + \frac{u^2+v^2}{a} + \frac{\mu}{\rho}\nabla^2 w,$
- $\frac{\partial T}{\partial t} = -\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]T + \frac{1}{c_v}\frac{dQ}{dt} - (\gamma - 1)T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$
- $\frac{\partial \rho}{\partial t} = -\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]\rho - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),\ \frac{\partial q}{\partial t} = -\left[u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right]q$
- $p = \rho RT$
- $\Longrightarrow$ Governing equations are non-linear partial differential equation

- These non-linear PDE can't be solved analytically, a couple of the reasons for which are:
  - We don't have any analytical expression for the time & space variations of the Meteorological variables.
  - Rather we have their numerical values at discrete points in space at a given time
  - Coefficients of the non-linear terms also don't have known analytical expression.
- Then what?
- Proceed for alternative approach – Numerical methods is one of the alternative approaches for time integration of the model equations.

Numerical methods

- In numerical method first the continuous time and 3-D space domain are discretized, like, $\{(x, y, z): (x, y, z) \in R^3\} \rightarrow \{(i\Delta x, j\Delta y, k\Delta z): (i, j, k) \in \mathbb{Z}^3 \& \Delta x, \Delta y, \Delta z \ given\ \}$ and time domain $\{t: 0 \ll t < \infty\} \rightarrow \{n\Delta t: n \in \mathbb{Z}\ \& \Delta t\ given\ \}$.
- The discrete spatial points $(i\Delta x, j\Delta y, k\Delta z)$ are denoted by (i, j, k) and called (i, j, k) grid point. Similarly, the discrete time $n\Delta t$ is called 'n' th time step.
- In numerical method values of the field variables (u, v, w, p, T, q, ρ) are specified at all discrete grid points at the time step '0' (initial time).
- Using these values of the field variables at different grid points at a given time step, spatial derivatives of the field variables are approximated numerically using a suitable finite difference scheme (FDS), for specifying the right-hand sides of the equations completely.
- This is followed by numerical integration in time for predicting values of the variable valid at next time step.

…..Numerical methods

- Finite difference methods:
  - To approximate numerically the time & space derivatives of the variables
  - Major finite differencing techniques:
    - Forward
    - Backward
    - Central or leapfrog

- Taylor's series: It is known that if a real valued function $f(x)$ is infinitely differentiable over the closed interval $[a, a+h]$, i.e., if $f(x)$ is analytical over $[a, a+h]$, then

$f(a+h) = f(a) + \sum_{n=1}^{\infty} \frac{h^n}{n!} f^{(n)}(a)$. Meteorological variables are assumed to be continuous in space & time domain.

- *Thus in a given grid* $[x_{j,} x_{j+1}]$, $f_{j\pm1}^{n} = f(x_{j\pm1}, n\Delta t) = f(x_j \pm \Delta x, n\Delta t) = f(x_j, n\Delta t) \pm \sum_{l=1}^{\infty} \frac{(\Delta x)^l}{l!} \left(\frac{\partial^l f}{\partial x^l}\right)_j^n = f_j^n \pm \sum_{l=1}^{\infty} \frac{(\Delta x)^l}{l!} \left(\frac{\partial^l f}{\partial x^l}\right)_j^n$

And $f_j^{n\pm1} = f_j^n \pm \sum_{k=1}^{\infty} \frac{(\Delta t)^k}{k!} \left(\frac{\partial^k f}{\partial t^k}\right)_j^n$

…...Numerical methods

Forward differencing scheme (FDS):

- $\left(\frac{\partial f}{\partial t}\right)_{(i,j,k)}^{n} \approx \frac{{f_{ijk}}^{n+1} - {f_{ijk}}^{n}}{\Delta t} + Terms\ multiple\ of\ \Delta t,$
- $\left(\frac{\partial f}{\partial x}\right)_{(i,j,k)}^{n} \approx \frac{{f_{(i+1)jk}}^{n} - {f_{ijk}}^{n}}{\Delta x} + Terms\ multiple\ of\ \Delta x$ etc.
- Error$\sim O(\Delta x, \Delta y, \Delta z, \Delta t)$

…..Numerical methods

Backward differencing scheme (BDS):

- $\left(\frac{\partial f}{\partial t}\right)_{(i,j,k)}^{n} \approx \frac{{f_{ijk}}^{n} - {f_{ijk}}^{n-1}}{\Delta t} + Terms\ multiple\ of\ \Delta t,$
- $\left(\frac{\partial f}{\partial x}\right)_{(i,j,k)}^{n} \approx \frac{{f_{ijk}}^{n} - {f_{(i-1)jk}}^{n}}{\Delta x} + Terms\ multiple\ of\ \Delta x$ etc.
- Error$\sim O(\Delta x, \Delta y, \Delta z, \Delta t)$

…..Numerical methods

Central differencing scheme or leap frog scheme (LFS):

- $\left(\frac{\partial f}{\partial t}\right)^{n}_{(i,j,k)} \approx \frac{f_{ijk}^{(n+1)} - f_{ijk}^{(n-1)}}{2\Delta t} + Terms\ multiple\ of\,(\Delta t)^2,$
- $\left(\frac{\partial f}{\partial x}\right)^{n}_{(i,j,k)} \approx \frac{f_{(i+1)jk}^{n} - f_{(i-1)jk}^{n}}{\Delta x} + Terms\ multiple\ of\,(\Delta x)^2$ etc.
- Error $\sim O[(\Delta x)^2, (\Delta y)^2, (\Delta z)^2, (\Delta t)^2]$

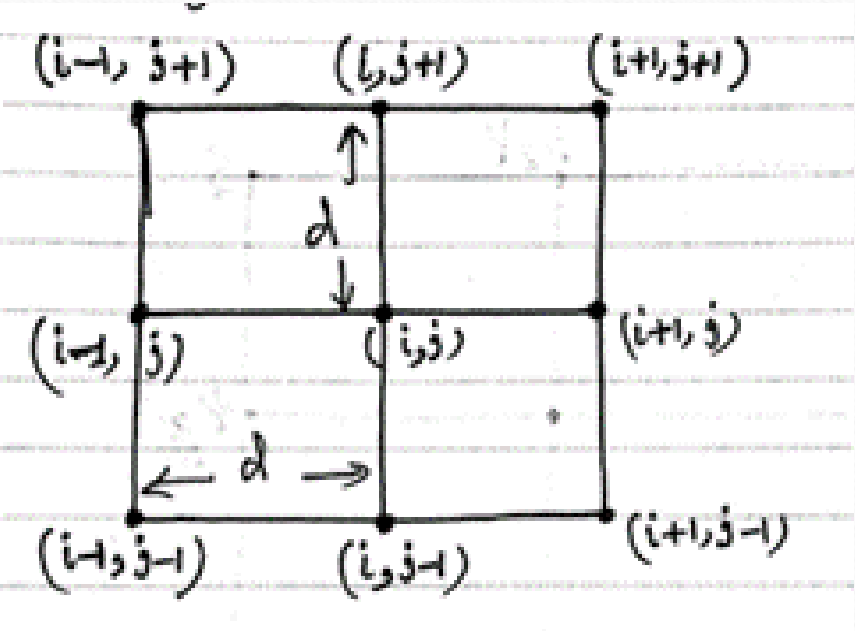
- Non linear horizontal advection of a scalar $S(x,y)$ can be expressed as $-\overrightarrow{V_H}.\overrightarrow{\nabla_H S}$
- $= -\left(u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y}\right) = -\left(-\frac{\partial \psi}{\partial y}\frac{\partial S}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial S}{\partial y}\right)..(1)$

  $= J(S,\psi); where\ \psi\ is\ a\ stream\ function\ \&J(S,\psi)$

Is the Jacobean of ψ and S. $J(S,\psi)$ can also be expressed as given below:

$J(S,\psi) = \frac{\partial}{\partial x}\left(\psi\frac{\partial S}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi\frac{\partial S}{\partial x}\right)\ldots(2)$ and

$J(S,\psi) = \frac{\partial}{\partial y}\left(S\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(S\frac{\partial \psi}{\partial y}\right)\ldots.(3)$



Arakawa 9-point Grid

Numerical approximation of Jacobean

- Numerical approximate value of the 3 expressions of $J(\psi, S) can\ be\ expressed\ as\ follows$:

- $$\left[\frac{\left(\psi_{(i+1,j)}-\psi_{(i-1,j)}\right)\left(S_{(i,j+1)}-S_{(i,j-1)}\right) - \left(S_{(i+1,j)}-S_{(i-1,j)}\right)\left(\psi_{(i,j+1)}-\psi_{(i,j-1)}\right)}{4d^2}\right] = J_1 \ldots (1)$$

- $$\left[\frac{\left\{\psi_{(i+1,j)}\left(S_{(i+1,j+1)}-S_{(i+1,j-1)}\right)-\psi_{(i-1,j)}\left(S_{(i-1,j+1)}-S_{(i-1,j-1)}\right)\right\} - \left\{\psi_{(i,j+1)}\left(S_{(i+1,j+1)}-S_{(i-1,j+1)}\right)-\psi_{(i,j-1)}\left(S_{(i+1,j-1)}-S_{(i-1,j-1)}\right)\right\}}{4d^2}\right] = J_2 \ldots (2)$$

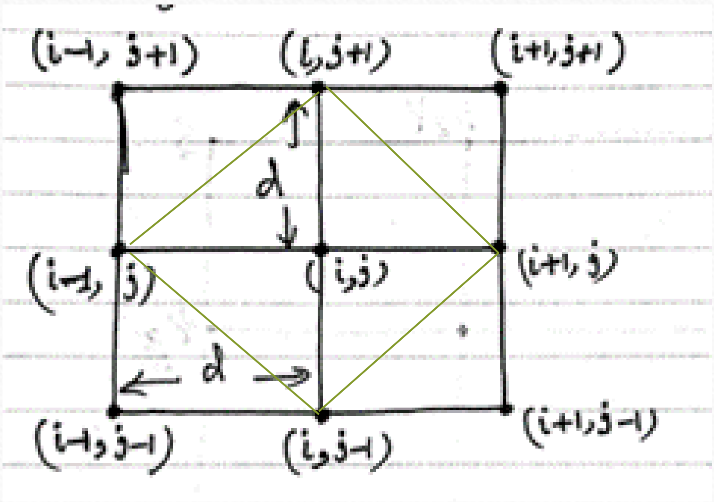
- $$\left[\frac{\left\{S_{(i,j+1)}\left(\psi_{(i+1,j+1)}-\psi_{(i-1,j+1)}\right)-S_{(i,j-1)}\left(\psi_{(i+1,j-1)}-\psi_{(i-1,j-1)}\right)\right\} - \left\{S_{(i+1,j)}\left(\psi_{(i+1,j+1)}-\psi_{(i+1,j-1)}\right)-S_{(i-1,j)}\left(\psi_{(i-1,j+1)}-\psi_{(i-1,j-1)}\right)\right\}}{4d^2}\right] = J_3 \ldots (3)$$

Numerical approximation of Laplacian

Laplacian of a scalar field $f(x, y)$ at any point (x, y) is given by,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

- $f_{i\pm1,j} = f(x_{i\pm1}, y_j) = f(x_i \pm \Delta x, y_j) = f(x_i, y_j) \pm \sum_{l=1}^{\infty} \frac{(\Delta x)^l}{l!} \left(\frac{\partial^l f}{\partial x^l}\right)_{i,j}$
- $f_{i,j\pm1} = f(x_i, y_{j\pm1}) = f(x_i, y_j + \Delta y) = f(x_i, y_j) \pm \sum_{l=1}^{\infty} \frac{(\Delta y)^l}{l!} \left(\frac{\partial^l f}{\partial y^l}\right)_{i,j}$



- Then, $(\nabla^2 f)_{i,j} \approx \frac{f_{(i+1,j)} + f_{(i-1,j)} + f_{(i,j+1)} + f_{(i,j-1)} - 4f_{(i,j)}}{d^2}$; where $\Delta x = \Delta y = d$ is the grid length.

Relaxation method for solving Poison's equation

General form of 2nd order PDE is: $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu$=G

A,B,C,D,E,F are coefficients, may be all of them constants or a couple/all of them may be functions of x,y; G is a known quantity may be a constant or a function of x,y and u(x,y) is an unknown function to be determined.

Above equation is called Parabolic, if $B^2 - 4AC = 0$

Elliptic, if $B^2 - 4AC < 0$

and Hyperbolic, if $B^2 - 4AC > 0$

Poison's equation is given by $\nabla^2 u = G(x, y)$. For this equation, $A = B = 1; C = D = E = F = 0$.

So, for this equation, $B^2 - 4AC = -4 < 0 \Rightarrow Poison's\ equation\ is\ an\ elliptic\ PDE.$

- Numerically approximate form of the above equation at a grid point (i, j) is

$$\frac{u_{(i+1,j)} + u_{(i-1,j)} + u_{(i,j+1)} + u_{(i,j-1)} - 4u_{(i,j)}}{d^2} = G_{(i,j)}$$

- This method starts with some initial guess values of the unknown function u(x,y) at all grid points. If, $u_{(i,j)}^{(0)}$ is the initial guess value of u(x,y) at any arbitrary grid point (i,j); then error in the initial guess, when substituted in the above equation, is given by

$$R_{(i,j)}^{(0)} = \frac{u_{(i+1,j)}^{(0)} + u_{(i-1,j)}^{(0)} + u_{(i,j+1)}^{(0)} + u_{(i,j-1)}^{(0)} - 4u_{(i,j)}^{(0)}}{d^2} - G_{(i,j)}$$

Above relation gives an improved guess value of u(x,y) at a grid point (i,j)

- $u_{(i,j)}^{(1)} = \frac{d^2}{4} R_{(i,j)}^{(0)} + u_{(i,j)}^{(0)}$

- Then, following similar arguments, the error in the first improved guess is given by

$$R_{(i,j)}^{(1)} = \frac{u_{(i+1,j)}^{(1)} + u_{(i-1,j)}^{(1)} + u_{(i,j+1)}^{(1)} + u_{(i,j-1)}^{(1)} - 4u_{(i,j)}^{(1)}}{d^2} - G_{(i,j)}$$

- And subsequently the second improved guess value is obtained as

- $u_{(i,j)}^{(2)} = \frac{d^2}{4} R_{(i,j)}^{(1)} + u_{(i,j)}^{(1)}$

The iteration process is said to converges when two successive improved guess of the unknown function u(x,y) differs by a number smaller than a very small pre-assigned positive number, say, ε,i.e., when $\left|u_{(i,j)}^{(m+1)} - u_{(i,j)}^{(m)}\right| < \varepsilon$, at every grid point (i,j).

Then either of these two successive improved guess value may be treated as approximate numerical solution of Poison's equation at a grid point (i,j).

Using this method, knowing horizontal wind components (u,v) at different grid points, one can find out stream function (ψ), velocity potential (χ), rotational wind $\left(\overrightarrow{V_\psi}\right)$ and divergent wind $\left(\overrightarrow{V_\chi}\right)$, using following steps:

Application

Vorticity(ς): $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \approx \left[\frac{v^n_{(i+1)jk} - v^n_{(i-1)jk}}{2\Delta x}\right] - \left[\frac{u^n_{i(j+1)k} - u^n_{i(j-1)k}}{2\Delta y}\right]$

Divergence $(D_h) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \left[\frac{u^n_{(i+1)jk} - v^n_{(i-1)jk}}{2\Delta x}\right] + \left[\frac{v^n_{i(j+1)k} - v^n_{i(j-1)k}}{2\Delta y}\right]$

- Set up the poison's equations for the stream function (ψ) and velocity potential (χ) : $\nabla^2\psi = \zeta(x, y)$ and $\nabla^2\chi = -D_h(x, y)$.
- Solve them using Relaxation method to find out ψ, χ at each grid point (i,j) at any vertical level 'k'.
- Then, rotational & divergent wind at any grid point are obtained as:
- $V_\psi = \hat{\imath}\left(-\frac{\partial\psi}{\partial y}\right) + \hat{\jmath}\left(\frac{\partial\psi}{\partial x}\right) \approx \hat{\imath}\left(-\left[\frac{\psi^n_{i(j+1)k} - \psi^n_{i(j-1)k}}{2\Delta y}\right]\right) + \hat{\jmath}\left[\frac{\psi^n_{(i+1)jk} - \psi^n_{(i-1)jk}}{2\Delta x}\right]$ and
- $V_\chi = -\left[\hat{\imath}\left(\frac{\partial\chi}{\partial x}\right) + \hat{\jmath}\left(\frac{\partial\chi}{\partial y}\right)\right] \approx -\left\{\hat{\imath}\left[\frac{\chi^n_{(i+1)jk} - \chi^n_{(i-1)jk}}{2\Delta x}\right] + \hat{\jmath}\left[\frac{\chi^n_{i(j+1)k} - \chi^n_{i(j-1)k}}{2\Delta y}\right]\right\}$

A few important concepts about Finite Difference Scheme

- Consistency or compatibility of a FDS: A FDS is said to be compatible or consistent if the FD approximation of derivative tends to its exact value or analytical value at each point / at each time as $\Delta x, \Delta y, \Delta z, \Delta t \to 0$.
- Convergence: Numerical solution of a well posed IVP is said to be convergence if it tends to analytical or exact solution as $\Delta x, \Delta y, \Delta z, \Delta t \to 0$
- Lax equivalence theorem: Given a well posed IVP and a consistent FDS; then numerical solution is convergent if and only if it is stable, i.e., as number of time step $(n) \to \infty$, at each point.

Explicit & implicit difference scheme

- To understand the concept of implicitness or explicitness of a differencing scheme, we refer the linear advection equation, viz., $\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial x}$, with c as constant phase speed.
- If the above equation is approximated numerically at a discrete time step 'n' and at a discrete spatial grid $'i'$, using forward and leap frog schemes, we get,

Forward difference scheme: $\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\mathrm{c}\frac{u_{i+1}^n - u_i^n}{\Delta x} \Rightarrow u_i^{n+1} = f(u_i^n, u_{i+1}^n)$

Central difference scheme: $\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -\mathrm{c}\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \Rightarrow u_i^{n+1} = f(u_i^{n-1}, u_{i+1}^n, u_{i-1}^n)$

- In both the above schemes, values at future time step is obtained using the values at present and/or past time steps. Such scheme is known as explicit scheme.

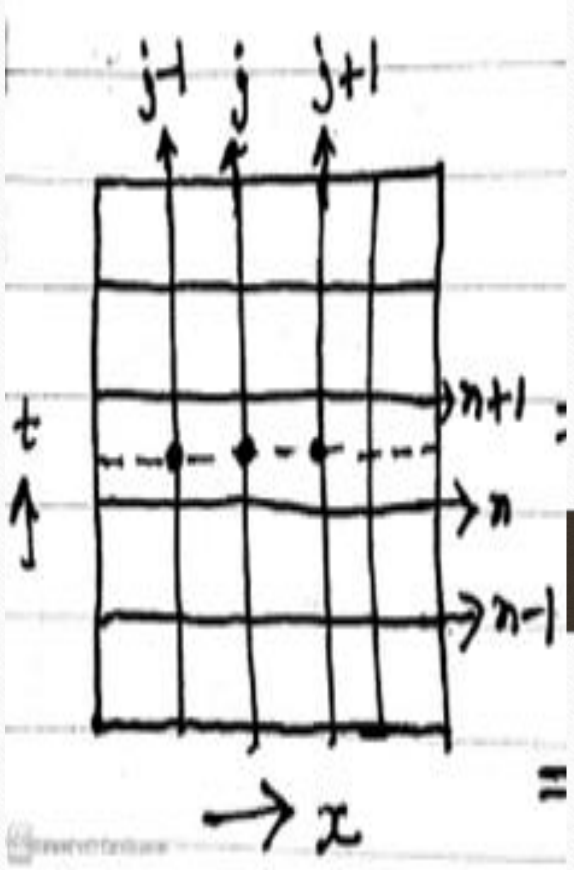
….Explicit & implicit difference scheme

- Time derivative is approximated numerically using forward difference scheme and space derivative is approximated using central difference scheme, averaged between time steps ‘n’ & ‘(n+1)’, as follows:

- $$\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\mathrm{c}\left[\frac{\frac{\left(u_{i+1}^{n+1}+u_{i+1}^n\right)}{2}-\frac{\left(u_{i-1}^{n+1}+u_{i-1}^n\right)}{2}}{2\Delta x}\right]$$

$$\Rightarrow u_i^{n+1} = f\left(u_i^n, u_{i+1}^n, u_{i-1}^{n}, u_{i+1}^{n+1}, u_{i-1}^{n+1}\right)$$

- Thus, value of the variable at a grid point at future time step (n+1) requires present value of the variable at the grid point and future value at neighbouring grid points.
- Such scheme is known as implicit scheme.



Issues with numerical methods- Linear computational instability-CFL criteria

- Solve the linear advection equation: $\frac{\partial f}{\partial t} = -c\frac{\partial f}{\partial x}$.

  Given, $f(x,0) = Ae^{ikx}$, c is constant phase speed.
- Its analytical/exact solution is $f(x;t) = Ae^{ik(x-ct)}$, a bounded solution.
- However, when attempted to solve numerically using LFS, it can be shown that the numerical solution is stable if $c\frac{\Delta t}{\Delta x} < 1$, otherwise unstable.
- Thus computational stability for LFS is conditional only

CFL criteria

Numerical solution of linear advection equation using LFS:

- $u_j^n = B^{n\Delta t} \exp(ikj\Delta x) \Rightarrow$
  $substituting\ in\ the\ LAE\ at\ ith\ grid\ \&\ nth\ time\ step, we\ obtain$

$B^{\Delta t} - B^{-\Delta t} = 2i\left(c\frac{\Delta t}{\Delta x}Sin(k\Delta x)\right) \Rightarrow B^{\Delta t} = \pm\sqrt{1-\sigma^2} + i\sigma, where\ \sigma =$
$c\frac{\Delta t}{\Delta x}Sin(k\Delta x)$. If $\sigma > 1, then\ magnitude\ of\ one\ of\ the\ solutions\ exceeds\ 1$

$$\Rightarrow B^{n\Delta t}\ becomes\ large\ for\ large\ 'n'.$$

Thus numerical solution is stable if $\sigma < 1 \Rightarrow c\frac{\Delta t}{\Delta x} < 1$. This is known as CFL criteria.

Thus LFS is conditionally stable.

Physical interpretation of CFL criteria

- Let us consider two successive grid points $i\Delta x$ & $(i+1)\Delta x$.
- Suppose there is an error caused at the grid point $i\Delta x$ *and the error propagates forward at a speed 'c'*.
- Then in one time step integration, the error can propagate a distance $c\Delta t$ forward.
- Thus to ensure that the error can't reach the next grid point $(i+1)\Delta x$, in one time integration to contaminate this grid point by the error, we should have, $c\Delta t < \Delta x \Rightarrow c\frac{\Delta t}{\Delta x} < 1 \Rightarrow$ *Physical interpretation of CFL criteria*.

Stability using semi implicit scheme

Numerical solution of linear advection equation using semi implicit scheme

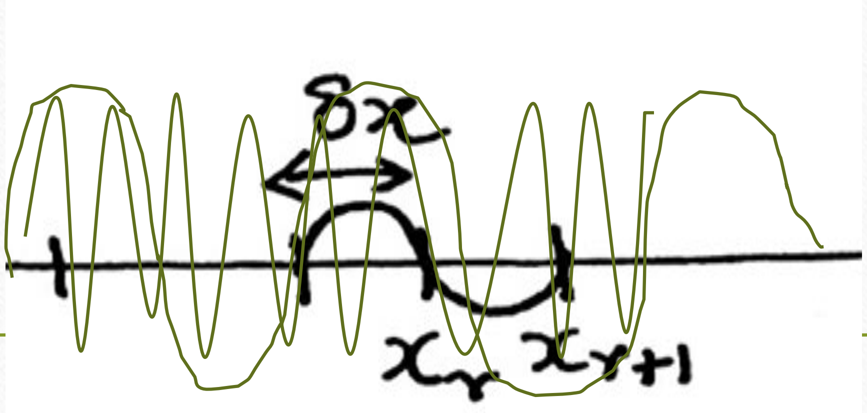
$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = -\mathrm{c}\left[\frac{\frac{\left(u_{j+1}^{n+1}+u_{j+1}^{n}\right)}{2}-\frac{\left(u_{j-1}^{n+1}+u_{j-1}^{n}\right)}{2}}{2\Delta x}\right]$$

- $u_j^n = B^{n\Delta t}\exp(ikj\Delta x)$
- $substituting\ in\ the\ LAE\ at\ ith\ grid\ \&\ nth\ time\ step, we\ obtain$
- $\left(B^{\Delta t}-1\right) = -ic\frac{\Delta t}{2\Delta x}\sin(k\Delta x)\left[\left(B^{\Delta t}+1\right)\right] \Rightarrow \frac{\left(B^{\Delta t}-1\right)}{\left(B^{\Delta t}+1\right)} = -\frac{i\sigma\sin(\mu\Delta x)}{2} \Rightarrow B^{\Delta t} = \frac{2-i\sigma\sin(\mu\Delta x)}{2+i\sigma\sin(\mu\Delta x)} == \frac{4+\sigma^2\sin^2(\mu\Delta x)-4i\sigma\sin(\mu\Delta x)}{4+\sigma^2\sin^2(\mu\Delta x)} \Rightarrow \left|B^{\Delta t}\right| = 1$
- Thus, $\left|B^{\Delta t}\right|^n = 1\ for\ any\ time\ step\ \ 'n'$. Hence this scheme is unconditionally or absolutely stable.

Issues with numerical methods- Non-linear instability

- Consider nonlinear advection equation $\frac{\partial f}{\partial t} = -\mathrm{u}\frac{\partial f}{\partial x}$, u is a function of x,t.
- Let us consider a limited interval$[a, b]$ and be divided into 'N' equal segments, by inserting grid points, $a = x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n = b$, with width δx between two arbitrary consecutive points.
- then the wave length of shortest possible wave is $2\delta x,$ as shown in adjoining figure.
- Let the dependent variables be expressed as $u(x,t) = \sum_{k=1}^{n} u1_k \cos kx + \sum_{k=1}^{n-1} u2_k \sin kx$ and
- $f(x,t) = \sum_{k=1}^{n} f1_k \cos kx + \sum_{k=1}^{n-1} f2_k \sin kx$
- Then the product term will have term like $\sin(m+l)\, x, \cos(m+l)\, x\ etc.$

- For some terms, $(m + l) > \frac{N}{2}$.
- Such terms corresponds to wave with wave length $< 2\delta x$.
- But the shortest wave, that can be represented with given grid arrangement is $2\delta x$.
- Thus a wave with wave length shorter than $2\delta x$ will be falsely represented by a relatively longer wave of wave length $2\delta x$.



- This false representation of a shorter wave by a longer wave is known as aliasing.
- Repeated aliasing gives rise to non linear instability.
- It is due to the presence of non linear term $u\frac{\partial f}{\partial x}$

Advection of a scalar field S can be expressed as $J(\psi, S), \psi$ being a stream function, related

with horizontal wind vector $\overrightarrow{V_H}\ as\ \overrightarrow{V_H} = \vec{k} X \vec{\nabla} \psi$.

It can be shown that if different expressions of J are numerically approximated at $(\mathrm{i}, \mathrm{j})\mathrm{th}$ grid point, numerically by say, $\mathrm{J}_1, \mathrm{J}_2$ &J_3; then Arakawa Jacobian, defined by $\mathrm{J} = \frac{\mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3}{3}$. If the advection term is numerically approximated by Arakawa Jacobian, then this Aliasing and non-linear instability can be eliminated.

- The governing equation for a non-divergent Barotropic model is

$\frac{d_h(\zeta+f)}{dt} = 0.$ In this model globally averaged ensthropy

$(\overline{\zeta^2})$ *and kinetic energy remains conserved.*

It is shown that in this model if the horizontal advection of vorticity is approximated either by J_1 or J_2 or J_3; *then both of* averaged ensthropy $(\overline{\zeta^2})$ *and kinetic energy don't remain conserved.*

However when the Jacobean $J(S,\psi)$ is numerically approximated by $\frac{J_1+J_2+J_3}{3}$, then it has been seen that both $(\overline{\zeta^2})$ *and kinetic energy remains conserved.* This ensures no Aliasing, thus non-linear instability is eliminated.

Thanks for kind & patience hearing