2.4. Radar Equation

The fundamental relation between the characteristics of the radar, the target and the received signal is called the radar equation and the theory of radar is developed based on that equation.

$$\mathbf{Pr} = \frac{PtG^2\theta^2 H\Pi^3 K^2 L}{1024(\ln 2)\lambda^2} x \frac{Z}{R^2}$$

This equation involves variables that are either known or are directly measured. There is only one value, **Pr,** that is missing but it can be solved for mathematically. Below is the list of variables, what they are and how they are measured.

Pr: Average power returned to the radar from a target. The radar sends pulses and then measures the average power that is received in those returns. The radar uses multiple pulses since the power returned by a meteorological target varies from pulse to pulse. This is an unknown value of the radar but it is one that is directly calculated.

Pt: Peak power transmitted by the radar. This is a known value of the radar. It is important to know because the average power returned is directly related to the transmitted power.

G: Antenna gain of the radar. This is a known value of the radar. This is a measure of the antenna's ability to focus outgoing energy into the beam. The power received from a given target is directly related to the square of the antenna gain.

θ: Angular beam width of radar. This is a known value of the radar. Through the Robert-Jones equation it can be learned that the return power is directly related to the square of the angular beam width. The problem becomes that the assumption of the equation is that precipitation fills the beam for radars with beams wider than two degrees. It is also an invalid assumption for any weather radar at long distances. The lower resolution at great distances is called the aspect ratio problem.

H: Pulse Length of the radar. This is a known value of the radar. The power received from a meteorological target is directly related to the pulse length.

K: This is a physical constant. This is a known value of the radar. This constant relies on the dielectric constant of water. This is an assumption that has to be made but also can cause some problems. The dielectric constant of water is near one, meaning it has a good reflectivity. The problem occurs when you have meteorological targets that do not share that reflectivity. Some examples of this are snow and dry hail since their constants are around 0.2.

L: This is the loss factor of the radar. This is a value that is calculated to compensate for attenuation by precipitation, atmospheric gases and receiver detection limitations. The attenuation by precipitation is a function of precipitation intensity and wavelength. For atmospheric gases, it is a function of elevation angle, range and wavelength. Since all of these accounts for a 2dB loss, all signals are strengthened by 2 dB.

λ: This is the wavelength of the transmitted energy. This is a known value of the radar. The amount of power returned from a precipitation target is inversely since the short wavelengths are subject to significant attenuation. The longer the wavelength, the less attenuation caused by precipitation.

Z: This is the reflectivity factor of the precipitate. This is the value that is solved for mathematically by the radar. The number of drops and the size of the drops affect this value. This value can cause problems because the radar cannot determine the size of the precipitate. The size is important since the reflectivity factor of a precipitation target is determined by raising each drop diameter in the sample volume to the sixth power and then summing all those values together. A ¼" drop reflects the same amount of energy as 64 1/8" drops even though there is 729 times more liquid in the 1/8" drops.

R: This is the target range of the precipitate. This value can be calculated by measuring the time it takes the signal to return. The range is important since the average power return from a target is inversely related to the square of its range from the radar. The radar has to normalize the power returned to compensate for the range attenuation.

Using a relationship between Z and R, an estimate of rainfall can be achieved. A base equation that can be used to do this is $Z=200*R^{1.6}$. This equation can be modified at the user's request to a better fitting equation for the day or the area.

2.4.1. How to derive radar equation?

It may be interesting for somebody so, a general derivation steps of radar equation is given below. Our starting point will be flux calculations.

Flux Calculations - Isotropic Transmit Antenna

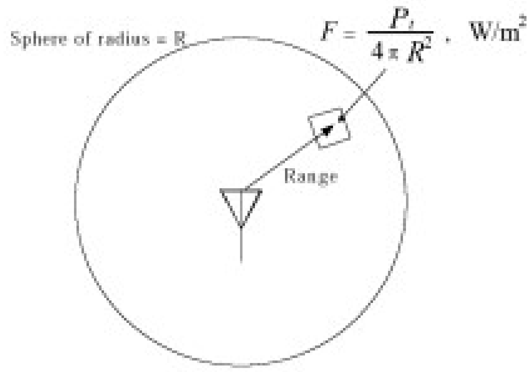

Figure 3: Flux at Distance R.

Flux Calculations - Transmit Antenna with Gain

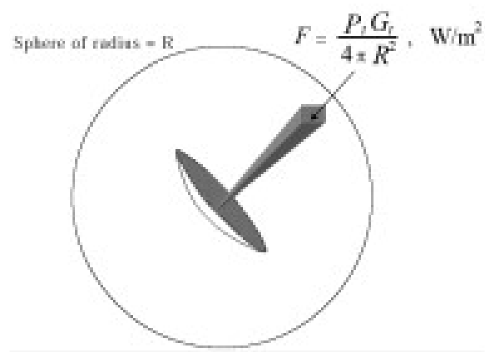

Figure 4: Flux at Distance R with Gain.

Radar Signal at Target, Incident power flux density from a Directive Source

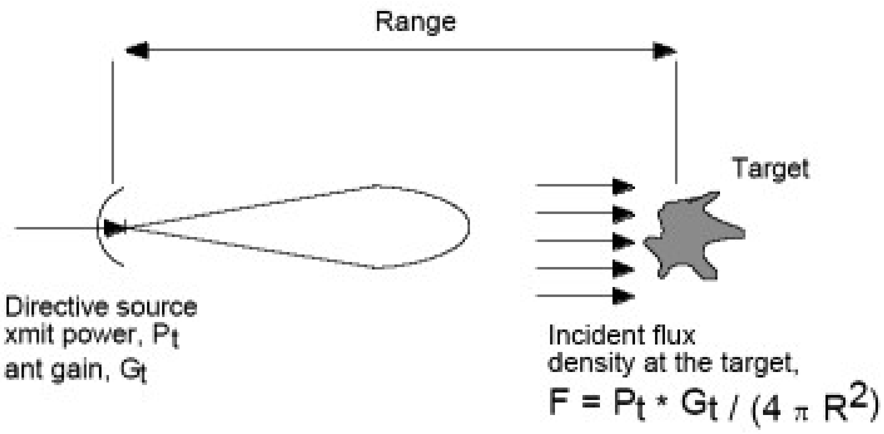

Figure 5: Incident Power Flux Density from a Directive Source.

Echo Signal at Target, Backscattered power from the target

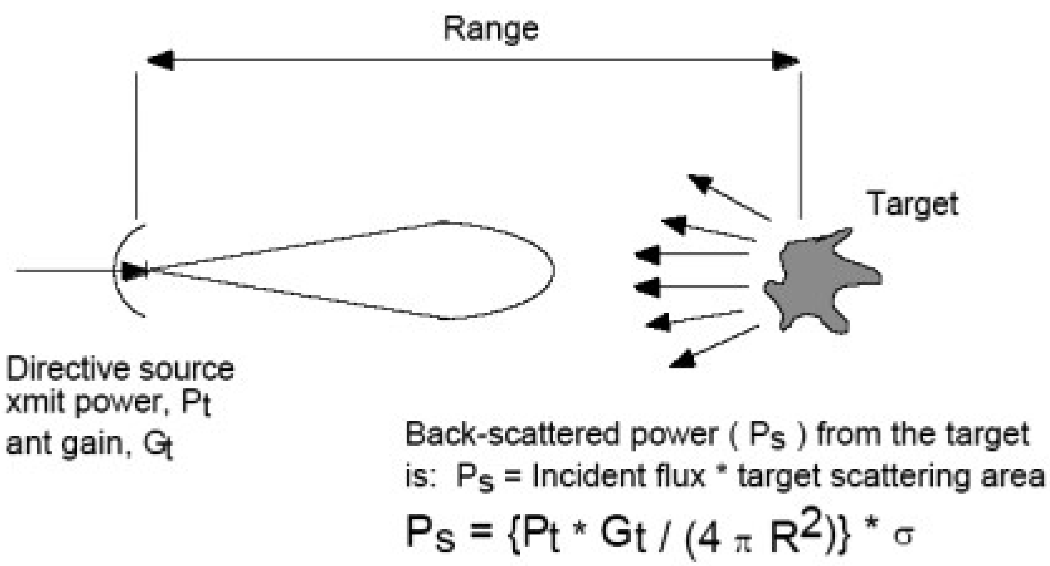

Figure 6: Power Back Scattered from Target with Cross Section.

Target Echo at Radar Backscattered power flux at the radar

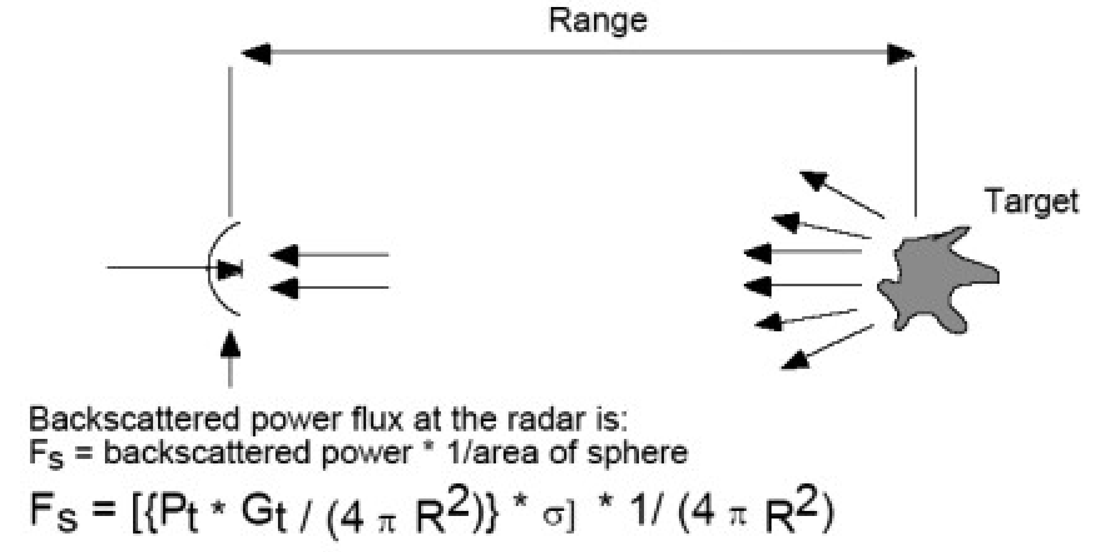

Figure 7: Flux Back Scattered from Target at Radar.

Radar Cross Section

The radar cross section (σ) of a target is the "equivalent area" of a flat-plate mirror:

- That is aligned perpendicular to the propagation direction (i.e., reflects the signal directly back to the transmitter) and
- That results in the same backscattered power as produced by the target

Radar cross section is extremely difficult to predict and is usually measured using scaled models of targets.

Target Echo Signal at Radar Received (echo) power at the radar

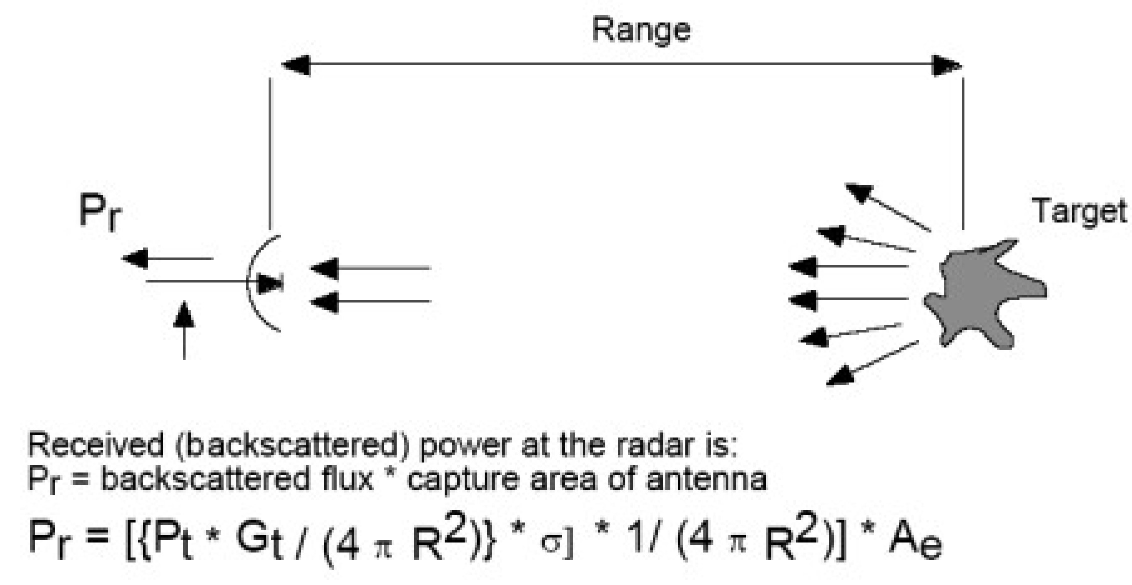

Figure 8: Received Power at Radar.

Relationship between Antenna Aperture and Gain

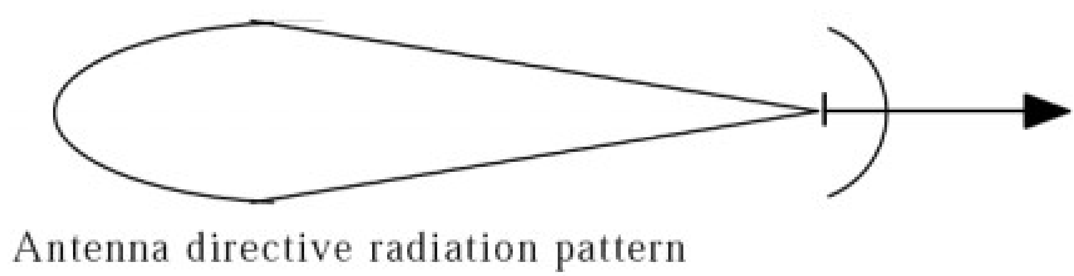

Figure 9: Antenna Aperture and Gain.

$$G_r = \frac{4\pi A_e}{\lambda^2}\text{, power ratio}$$

$$A_e = \rho_a * A\text{, } meters^2$$

Where A= the physical aperture area of the antenna

ρ_a = the aperture collection efficiency λ= wave length electromagnetic = c/freq

Idealized Radar Equation - no system losses

$$P_r = P_t * G_t * G_r \left[\frac{\lambda^2}{(4\pi)^3 R^4} \right] * \sigma \ , \ watts$$

Since the antenna gain is the same for transmit and receive, this becomes:

$$P_r = P_t * G^2 \left[\frac{\lambda^2}{(4\pi)^3 R^4} \right] * \sigma \ , \ watts$$

Practical Radar Equation - with system losses for point targets

$$P_r = P_t * G^2 \left[\frac{\lambda^2}{(4\pi)^3 R^4} \right] * \sigma * L_{sys} \ , \ watts$$

Where:

- Lsys is the system losses expressed as a power ratio,
- P_r is the average received power,
- P_t is the transmitted power,
- G is the gain for the radar,
- λ is the radar's wavelength,
- σ is the targets scattering cross section,
- R is the range from the radar to the target.

The radar equation for a point target is simply given below:

$$\overline{P}_r = \frac{P_t G^2 \lambda^2 \sigma}{(4\pi)^3 R^4}$$

Radar Equation for Distributed Targets

Thus far, we've derived the radar equation for a point target. This is enough if you are interested in point targets such as airplanes. However, in a thunderstorm or some area of precipitation, we do not have just one target (e.g., raindrop), we have many. Thus we need to derive the radar equation for distributed targets. So let's review the Radar Pulse Volume.

Radar Pulse Volume

First, let's simplify the real beam according to the **Figure 10**:

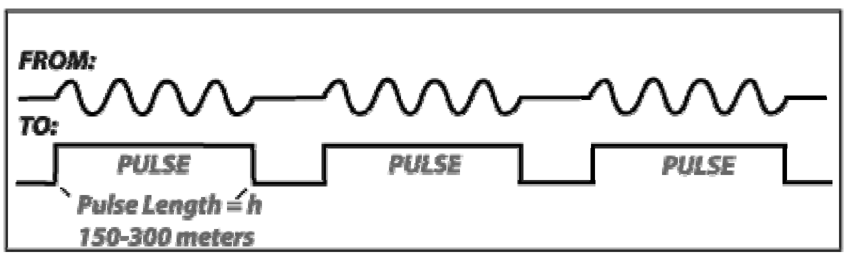

Figure 10: Radar Pulses.

What does a "three-dimensional" segment of the radar beam look like?

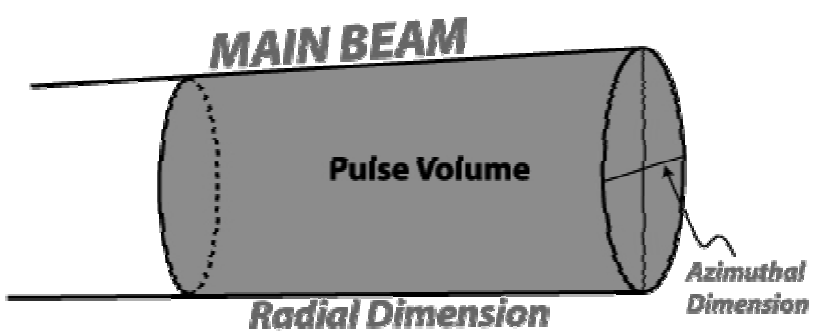

Figure 11: Radar Main Beam and Pulse Volume.

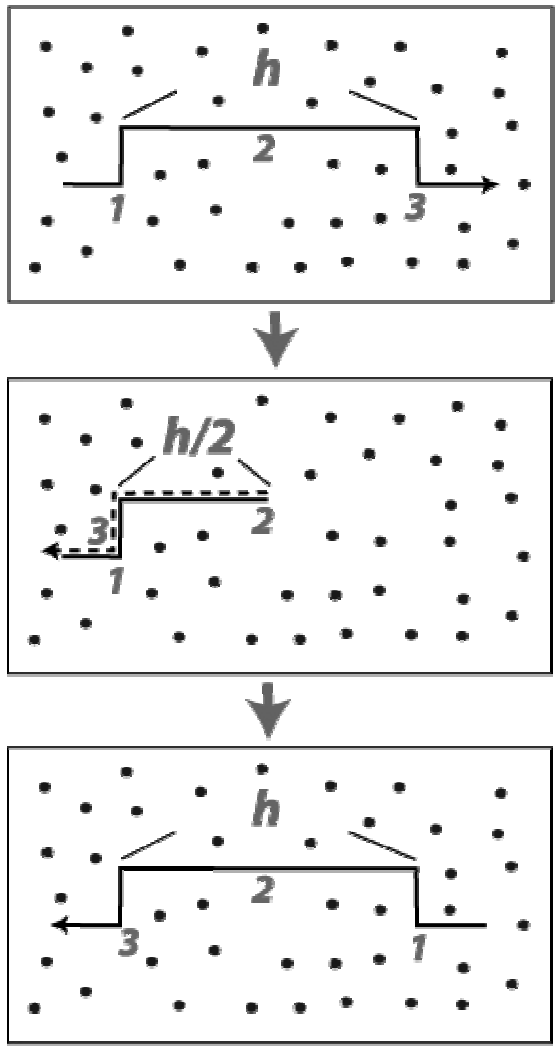

Figure 12: Form of Transmit and Received Signal.

So, the “volume” of the pulse volume is:

$$\pi \frac{R\theta}{2}\frac{R\phi}{2}\frac{h}{2}$$

For a circular beam, then θ=Φ, the pulse volume becomes:

$$\pi \frac{R^2\theta^2 h}{8}$$

Before we derive the radar equation for the distributed targets situation, we need to make some assumptions:

1) The beam is filled with targets.
2) Multiple scattering is ignored
3) Total average power is equal to sum of powers scattered by individual particles.

Recall the radar equation for a single target:

$$\overline{P}_r = \frac{P_t G^2 \lambda^2 \sigma}{(4\pi)^3 R^4} \quad (1)$$

For multiple targets, radar equation (1) can be written as:

$$\overline{P}_r = \frac{P_t G^2 \lambda^2}{(4\pi)^3} \sum_i \frac{\sigma_i}{R_i^4} \quad (2)$$

where the sum is over all targets within the pulse volume.

If we assume that *h/2 << r_i*,

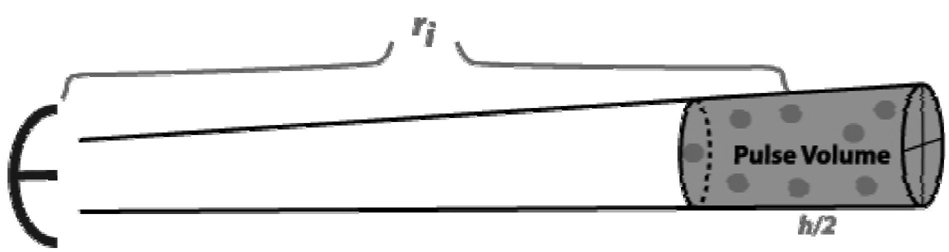

Figure 13: Pulse Volume.

Then (2) can be written as:

$$\overline{P}_r = \frac{P_t G^2 \lambda^2}{(4\pi)^3 R_i^4} \sum_i \sigma_i \quad (3)$$

It is advantageous to sum the backscattering cross sections over a unit volume of the total pulse volume.

Hence the sum in (3) can be written as:

$$\sum_i \sigma_i = \left(\sum_i \frac{\sigma_i}{unitvolume} \right) totalvolume \quad (4)$$

where the total volume is the volume of the pulse.

Thus, (5) can be written as:

$$\sum_i \sigma_i = \left(\sum_i \frac{\sigma_i}{unitvolume} \right) \pi \frac{R^2 \theta^2 h}{8} \quad (5)$$

Substituting (5) into (3) gives:

$$\overline{P}_r = \left(\frac{P_t G^2 \lambda^2 \theta^2 h}{512 \pi^2 R^2} \right) \sum_i \sigma_i \quad (6)$$

Note that:

P_r is proportional to R^{-2} for distributed targets.

P_r is proportional to R^{-4} for point targets.

Radar Reflectivity

The sum of all backscattering cross sections (per unit volume) is referred to as the radar reflectivity (η). In other words,

$$\sum_i \sigma_i = \eta \quad (7)$$

In terms of the radar reflectivity, the radar equation for distributed targets (21) can be written as:

$$\overline{P}_r = \left(\frac{P_t G^2 \lambda^2 \theta^2 h}{512 \pi^2 R^2} \right) \eta \quad (8)$$

All variables in (8), except η are either known or measured.

Now, we need to add a fudge factor due to the fact that the beam shape is Gaussian.

Hence, (8) becomes;

$$\overline{P}_r = \frac{1}{2 \ln 2} \left(\frac{P_t G^2 \lambda^2 \theta^2 h}{512 \pi^2 R^2} \right) \eta \quad (9)$$

Complex Dielectric Factor

The backscattering cross section (σ_i) can be written as:

$$\sigma = \frac{\pi^5 |K|^2 D^6}{\lambda^4} \qquad (10)$$

Where:

- D is the diameter of the target,
- λ is the wavelength of the radar,
- K is the complex dielectric factor,
  - σ is some indication of how good a material is at backscattering radiation

For water $|K|^2$=0.93

For ice $|K|^2$=0.197

Notice that the value for water is much larger than for ice. All other factors the same; this creates a 5dB difference in returned power.

So, let's incorporate this information into the radar equation.

Recall from (7) that. $\sum_i \sigma_i = \eta$ Using (11) can be written

As: $$\eta = \sum_i \sigma_i = \sum_i \frac{\pi^5 |K|^2 D_i^6}{\lambda^4} \qquad (12)$$

Taking the constants out of the sum;

$$\eta = \frac{\pi^5 |K|^2}{\lambda^4} \sum_i D_i^6 \qquad (13)$$

Remember that the sum is for a unit volume. Substituting (12) into (9) gives:

$$\overline{P_r} = \frac{1}{2\ln 2}\left(\frac{P_t G^2 \lambda^2 \theta^2 h}{512\pi^2 R^2}\right)\frac{\pi^5 |K|^2}{\lambda^4}\sum_i D_i^6 \qquad (14)$$

Simplifying terms gives:

$$\overline{P}_r = \frac{P_t G^2 \theta^2 \pi^3 h |K|^2}{1024 \ln 2 R^2 \lambda^2} \sum_i D_i^6$$

(15)

Note the D_i^6 dependence on the average received power.

Radar Reflectivity Factor

In Equation (15), all variables except the summation term, are either known or measured.

We will now define the radar reflectivity factor, Z as:

$$Z = \sum_i \frac{D_i^6}{unitvolume} \quad (16)$$

Substituting (30) into (29) gives the radar equation for distributed targets:

$$\overline{P}_r = \frac{P_t G^2 \theta^2 \pi^3 h |K|^2 Z}{1024 \ln 2 \lambda^2 R^2} \quad (17)$$

- Note the relationship between the received power, range and radar wavelength
- Everything in Equation (17) is measured or known except *Z*, the radar reflectivity factor.
- Since the strength of the received power can span many orders of magnitude, then so do Z.
- Hence, we take the log on Z according to:

$$dBZ = 10 \log\left(\frac{Z}{1}\frac{mm^6}{m^3}\right) \quad (18)$$

- The dBZ value calculated above is what you see displayed on the radar screen or on imagery accessed from the web.

As a result, formulas can be written as follows:

- **Point target radar equation**:

$$p_r = \frac{p_t g^2 \lambda^2 A_\sigma}{64\pi^3 r^4}$$

- **Meteorological target radar equation**

$$p_r = \frac{\pi^5 p_t g^2 \theta\phi c \tau |K|^2 zl}{1024 \ln(2) \lambda^2 r^2}$$